

POSITIVE PARTITION RELATIONS FOR $P_\kappa(\lambda)$

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Abstract. Let κ a regular uncountable cardinal and λ a cardinal $> \kappa$, and suppose $\lambda^{<\kappa}$ is less than the covering number for category $\text{cov}(\mathbf{M}_{\kappa,\kappa})$. Then (a) $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$, (b) $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_{\kappa^+}^2$ if κ is a limit cardinal, and (c) $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$ if κ is weakly compact.

0. Introduction

Let κ be a weakly compact cardinal. Then $\kappa \rightarrow (\kappa)^2$ and more generally for any cardinal $\lambda \geq \kappa$, $\{P_\kappa(\lambda)\} \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$ ([M4]), which means that for any $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$, there is $A \subseteq P_\kappa(\lambda)$ such that A does not belong to $I_{\kappa,\lambda}$ (the ideal of noncofinal subsets of $P_\kappa(\lambda)$) and F is constant on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}.$$

Now if J is the ideal of noncofinal subsets of κ , then $J^+ \rightarrow (J^+)^2$ since $(A, <)$ is isomorphic to $(\kappa, <)$ for any $A \in J^+$. So it is natural to ask whether $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$ for every $\lambda > \kappa$. It turns out that the answer is negative. This is not surprising since it is well-known that some members of $I_{\kappa,\lambda}^+$ may be quite different from $P_\kappa(\lambda)$. To give an example, if the GCH holds and λ is the successor of a cardinal of cofinality $< \kappa$, then $\overline{\text{cof}}(I_{\kappa,\lambda} \mid A) < \overline{\text{cof}}(I_{\kappa,\lambda})$ for some $A \in I_{\kappa,\lambda}^+$ ([MPéS2]). We prove that $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$ if and only if $\lambda^{<\kappa}$ is less than $\text{cov}(\mathbf{M}_{\kappa,\kappa})$ (a generalization of the covering number for category $\text{cov}(\mathbf{M})$).

Let κ be an arbitrary regular uncountable cardinal. Dushnik and Miller [DMi] established that $\kappa \rightarrow (\kappa, \omega)^2$. This was improved to $\kappa \rightarrow (\kappa, \omega + 1)^2$ by Erdős and Rado [ER]. The Erdős-Rado result generalizes ([M3]) : for every cardinal $\lambda \geq \kappa$, $\{P_\kappa(\lambda)\} \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$ (i.e. for any $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$, there is either $A \in I_{\kappa,\lambda}^+$ such that F is identically 0 on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\},$$

or $a_0, a_1, \dots, a_\omega$ in $P_\kappa(\lambda)$ such that $a_0 \subset a_1 \subset \dots \subset a_\omega$, $\cup(a_0 \cap \kappa) < \cup(a_1 \cap \kappa) < \dots < \cup(a_\omega \cap \kappa)$ and F is identically 1 on $\{(\cup(a_n \cap \kappa), a_q) : n < q \leq \omega\}$). Here we show that $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$ if $\lambda^{<\kappa}$ is less than $\text{cov}(\mathbf{M}_{\kappa,\kappa})$. In the other direction we prove that $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} (I_{\kappa,\lambda}^+, 3)^2$ if λ is greater than or equal to \mathfrak{d}_κ (or even $\overline{\mathfrak{d}}_\kappa$).

It is a result of [M5] that $\{P_\kappa(\lambda)\} \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$ for any $\lambda > \kappa$ if κ is a successor cardinal such that $\kappa \rightarrow [\kappa]_\kappa^2$. In contrast to this, we show that $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_{\kappa^+}^2$ if κ is a limit cardinal and λ a cardinal $> \kappa$ with $\lambda^{<\kappa} < \text{cov}(\mathbf{M}_{\kappa,\kappa})$. It is also shown that $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$ if $\lambda \geq \overline{\mathfrak{d}}_\kappa$.

Throughout the remainder of this paper κ will denote a regular uncountable cardinal and λ a cardinal $> \kappa$.

The paper is organized as follows. Section 1 reviews a number of standard definitions concerning ideals on κ and $P_\kappa(\lambda)$. Sections 2-7 give results about combinatorics on κ that are needed for our study of $P_\kappa(\lambda)$. Sections 2 and 3 review some facts concerning, respectively, the dominating number \mathfrak{d}_κ and the

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covering number for category $\text{cov}(\mathbf{M}_{\kappa,\kappa})$. Section 4 deals with the problem of determining the value of the inequality number \mathfrak{U}_κ in the case where κ is a successor cardinal. In Section 5 we show that if $2^{<\kappa} = \kappa$ and $\mathfrak{U}_\kappa < \kappa^{+\omega}$, then $\mathfrak{U}_\kappa = \text{non}_\kappa$ (weakly selective). Sections 6 and 7 review some material concerning, respectively, the unbalanced partition relation $J^+ \rightarrow (J^+, \rho)^2$ and the square bracket partition relation $J^+ \rightarrow [J^+]_\rho^2$.

Sections 8-15 are concerned with combinatorial properties of ideals on $P_\kappa(\lambda)$. Section 8 gives two characterizations of $\mathfrak{d}_{\kappa,\lambda}^\kappa$: one as the least cofinality of any κ -complete fine ideal on $P_\kappa(\lambda)$ that is not a weak π -point, and the other as the least cofinality of any κ -complete fine ideal on $P_\kappa(\lambda)$ that admits a maximal almost disjoint family of size κ . In Section 9 we show that any κ -complete fine ideal on $P_\kappa(\lambda)$ with cofinality $< \text{cov}(\mathbf{M}_{\kappa,\kappa})$ is a weak χ -point. Conversely if κ is inaccessible and $I_{\kappa,\lambda}$ is a weak χ -point, then $\text{cof}(I_{\kappa,\lambda}) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$. Sections 10-13 deal with unbalanced partition relations. Given an infinite cardinal $\theta \leq \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$, we show that (a) $u(\kappa, \lambda) \cdot \text{non}_\kappa(J^+ \rightarrow (J^+, \theta)^2)$ is the least cofinality of any κ -complete fine ideal H on $P_\kappa(\lambda)$ such that $H^+ \xrightarrow[\kappa]{} (H^+, \theta)^2$, (b) If H is a κ -complete fine ideal on $P_\kappa(\lambda)$ with $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$ (respectively, $\text{cof}(H) < \text{non}_\kappa$ (weakly selective)), then $H^+ \rightarrow (H^+, \theta)^2$ (respectively, $H^+ \xrightarrow[\kappa]{} (H^+, \theta)^2$), and (c) Conversely, if $\theta = \kappa$ and $I_{\kappa,\lambda}^+ \xrightarrow[\kappa]{} (H^+, \theta)^2$, then $\text{cof}(I_{\kappa,\lambda}) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$. The last two sections are concerned with square bracket partition relations. We show that if κ is a limit cardinal, then $H^+ \xrightarrow[\kappa]{} [H^+]_{\kappa^+}^2$ (respectively, $H^+ \xrightarrow[\kappa]{} [H^+]_\kappa^2$) for every ideal H on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$ (respectively, $\text{cof}(H) < \text{non}_\kappa(J^+ \rightarrow [J^+]_\kappa^2)$). In the other direction, $\lambda \geq \overline{\mathfrak{d}}_\kappa$ implies that $I_{\kappa,\lambda}^+ \not\xrightarrow[\kappa]{} [I_{\kappa,\lambda}^+]_\lambda^2$ (and $I_{\kappa,\lambda}^+ \not\xrightarrow[\kappa]{} [I_{\kappa,\lambda}^+]_\kappa^2$ if κ is a limit cardinal such that $2^{<\kappa} = \kappa$).

1. Ideals

In this section we review some standard definitions and a few basic facts concerning ideals on κ and $P_\kappa(\lambda)$.

Given a cardinal μ and a set A , let $P_\mu(A) = \{a \subseteq A : |a| < \mu\}$.

Given an infinite set S , an *ideal on S* is a collection K of subsets of S such that (i) $\{s\} \in K$ for every $s \in S$, (ii) $P(A) \subseteq K$ for every $A \in K$, (iii) $A \cup B \in K$ whenever $A, B \in K$, and (iv) $S \notin K$.

Given an ideal K on S , let $K^+ = P(S) - K$ and $K \mid A = \{B \subseteq S : B \cap A \in K\}$ for $A \in K^+$. $\text{sat}(K)$ is the least cardinal τ with the property that for every $Y \subseteq K^+$ with $|Y| = \tau$, there exist $A, B \in Y$ such that $A \neq B$ and $A \cap B \in K$.

$\text{cof}(K)$ is the least cardinality of any $X \subseteq K$ such that $K = \bigcup_{A \in X} P(A)$. K is κ -complete if $\bigcup X \in K$ for every $X \in P_\kappa(K)$. Assuming that K is κ -complete and $\bigcup Y \in K^+$ for some $Y \subseteq K$ with $|Y| = \kappa$, $\overline{\text{cof}}(K)$ is the least cardinality of any $X \subseteq K$ such that $K = \bigcup\{P(\bigcup x) : x \in P_\kappa(X)\}$.

We adopt the convention that the phrase “ideal on κ ” means “ κ -complete ideal on κ ”.

Note that the smallest ideal on κ is $P_\kappa(\kappa)$.

Given two sets A and B and $f \in {}^A B$, f is *regressive* if $f(a) \in a$ for all $a \in A$.

An ideal J on κ is *normal* if given $A \in J^+$ and a regressive $f \in {}^A \kappa$, there is $B \in J^+ \cap P(A)$ such that f is constant on B .

NS_κ denotes the nonstationary ideal on κ .

κ is *inaccessible* if $2^\mu < \kappa$ for every cardinal $\mu < \kappa$.

Let $[A]^2 = \{(\alpha, \beta) \in A \times A : \alpha < \beta\}$ for any $A \subseteq \kappa$. Given an ordinal $\alpha \geq 2$, $\kappa \rightarrow (\kappa, \alpha)^2$ means that for every $f : [\kappa]^2 \rightarrow 2$, there is $A \subseteq \kappa$ such that either A has order type κ and f is identically 0 on $[A]^2$, or A has order type α and f is identically 1 on $[A]^2$. The negation of this and other partition relations is indicated by crossing the arrow. $\kappa \rightarrow (\kappa)^2$ means that $\kappa \rightarrow (\kappa, \kappa)^2$.

κ is *weakly compact* if $\kappa \rightarrow (\kappa)^2$.

If κ is weakly compact, then it is inaccessible (see e.g. Proposition 4.4 in [Ka]).

An ideal J on κ is a *weak P-point* if given $A \in J^+$ and $f \in {}^A\kappa$ with $\{f^{-1}(\{\alpha\}) : \alpha < \kappa\} \subseteq J$, there is $B \in J^+ \cap P(A)$ such that f is $< \kappa$ -to-one on B . J is a *local Q-point* if given $g \in {}^\kappa\kappa$, there is $B \in J^+$ such that $g(\alpha) < \beta$ for any $(\alpha, \beta) \in [B]^2$. J is a *weak Q-point* if $J \mid A$ is a local Q-point for every $A \in J^+$.

It is well-known (see [M1] for a proof) that an ideal J on κ is a weak Q-point if and only if given $A \in J^+$ and a $< \kappa$ -to-one $f : A \rightarrow \kappa$, there is $B \in J^+ \cap P(A)$ such that f is one-to-one on B .

An ideal J on κ is *weakly selective* if it is both a weak P-point and a weak Q-point.

Given a cardinal ρ with $2 \leq \rho \leq \kappa$ and an ideal J on κ , $J^+ \rightarrow [J^+]_\rho^2$ means that for every $A \in J^+$ and every $f : [A]^2 \rightarrow \rho$, there is $B \in J^+ \cap P(A)$ such that $f''[B]^2 \neq \rho$. $\kappa \rightarrow [\kappa]_\rho^2$ means that $(P_\kappa(\kappa))^+ \rightarrow [(P_\kappa(\kappa))^+]_\rho^2$.

Note that $\kappa \rightarrow [\kappa]_2^2$ if and only if $\kappa \rightarrow (\kappa)^2$.

Let P be a property such that at least one ideal on κ does not satisfy P . Then $\text{non}_\kappa(P)$ (respectively, $\overline{\text{non}}_\kappa(P)$) denotes the least cardinal τ for which one can find an ideal J on κ such that $\text{cof}(J) = \tau$ (respectively, $\text{cof}(J) = \tau$) and J does not satisfy P .

Notice that $\lambda^{<\kappa} < \overline{\text{non}}_\kappa(P)$ if and only if $\lambda^{<\kappa} < \text{non}_\kappa(P)$.

$I_{\kappa, \lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $A \cap \{b \in P_\kappa(\lambda) : a \subseteq b\} = \emptyset$ for some $a \in P_\kappa(\lambda)$. An ideal H on $P_\kappa(\lambda)$ is *fine* if $I_{\kappa, \lambda} \subseteq H$.

We adopt the convention that the phrase “ideal on $P_\kappa(\lambda)$ ” means “ κ -complete fine ideal on $P_\kappa(\lambda)$ ”.

Note that $I_{\kappa, \lambda}$ is the smallest ideal on $P_\kappa(\lambda)$.

$u(\kappa, \lambda)$ denotes the least cardinality of any $A \in I_{\kappa, \lambda}^+$.

The following facts are well-known (see e.g. [MPéS1]) : (1) $u(\kappa, \lambda) \geq \lambda$; (2) $\lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda)$; (3) $u(\kappa, \lambda) = \text{cof}(I_{\kappa, \lambda} \mid A)$ for every $A \in I_{\kappa, \lambda}^+$; (4) $u(\kappa, \kappa^{+n}) = \kappa^{+n}$ whenever $0 < n < \omega$.

$\mathcal{K}(\kappa, \lambda)$ denotes the set of all cardinals $\sigma \geq \lambda$ with the property that there is $T \subseteq P_\kappa(\lambda)$ such that $|T| = \sigma$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

It is simple to see that $\sigma \leq u(\kappa, \lambda)$ for every $\sigma \in \mathcal{K}(\kappa, \lambda)$. Notice that $\lambda \in \mathcal{K}(\kappa, \lambda)$. More generally, if τ is an infinite cardinal $\leq \kappa$ such that $|P_\tau(\nu)| < \kappa$ for every infinite cardinal $\nu < \kappa$, then $\lambda^{<\tau} \in \mathcal{K}(\kappa, \lambda)$. It follows that $\lambda^{<\kappa} \in \mathcal{K}(\kappa, \lambda)$ if κ is inaccessible. It can be shown (see Remark 11.4 in [To 2] and Theorem 4.1 in [CFMag]) that $\lambda^+ \in \mathcal{K}(\kappa, \lambda)$ if \square_κ^* holds and $\text{cf}(\lambda) < \kappa$.

An ideal H on $P_\kappa(\lambda)$ is *κ -normal* if given $A \in H^+$ and a regressive $f \in {}^A\kappa$, there is $B \in H^+ \cap P(A)$ such that f is constant on B . The smallest κ -normal ideal on $P_\kappa(\lambda)$ is denoted by $NS_{\kappa, \lambda}^\kappa$.

2. Domination

In this section we recall some characterizations of the dominating number \mathfrak{d}_κ .

Definition. \mathfrak{d}_κ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in X$ such that $g(\alpha) < f(\alpha)$ for all $\alpha < \kappa$.

$\overline{\mathfrak{d}}_\kappa$ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $x \in P_\kappa(X)$ such that $g(\alpha) < \bigcup_{f \in x} f(\alpha)$ for all $\alpha < \kappa$.

PROPOSITION 2.1.

- (i) ([L1]) $\mathfrak{d}_\kappa = \text{cof}(NS_\kappa)$.
- (ii) ([MRoS]) $\overline{\mathfrak{d}}_\kappa = \overline{\text{cof}}(NS_\kappa)$.

Definition. Given an ideal J on κ , $\mathcal{M}_J^{\geq\kappa}$ is the set of all $Q \subseteq J^+$ such that (i) $|Q| \geq \kappa$, (ii) $A \cap B \in J$ for all $A, B \in Q$ with $A \neq B$, and (iii) for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.

\mathfrak{a}_J is the least cardinality of any member of $\mathcal{M}_J^{\geq\kappa}$ if $\mathcal{M}_J^{\geq\kappa} \neq \emptyset$, and $(2^\kappa)^+$ otherwise.

THEOREM 2.2. ([Laf], [MP2]) $\mathfrak{d}_\kappa = \text{non}_\kappa(\mathfrak{a}_J > \kappa) = \text{non}_\kappa(\text{weak } P\text{-point})$.

PROPOSITION 2.3. $\overline{\mathfrak{d}}_\kappa \geq \overline{\text{non}}_\kappa(\mathfrak{a}_J > \kappa) \geq \overline{\text{non}}_\kappa(\text{weak } P\text{-point})$.

Proof. The first inequality follows from Proposition 2.1 (ii) since $\mathfrak{a}_{NS_\kappa} = \kappa$ ([MP2]). To prove the second inequality, argue as for Lemma 8.5 below. \square

QUESTION. Is it consistent that $\overline{\mathfrak{d}}_\kappa > \overline{\text{non}}_\kappa(\text{weak } P\text{-point})$?

3. Covering for category

Throughout this section ν will denote a fixed regular infinite cardinal.

We will review some basic facts concerning the covering number $\text{cov}(\mathbf{M}_{\nu,\nu})$.

Definition. Suppose ρ is a cardinal $\geq \nu$.

Let $Fn(\rho, 2, \nu) = \cup\{{}^a 2 : a \in P_\nu(\rho)\}$. $Fn(\rho, 2, \nu)$ is ordered by : $p \leq q$ if and only if $q \subseteq p$.

${}^\rho 2$ is endowed with the topology obtained by taking as basic open sets ϕ and O_s^ρ for $s \in Fn(\rho, 2, \nu)$, where $O_s^\rho = \{f \in {}^\rho 2 : s \subseteq f\}$.

$\mathbf{M}_{\nu,\rho}$ is the set of all $W \subseteq {}^\rho 2$ such that $W \cap (\cap X) = \phi$ for some collection X of dense open subsets of ${}^\rho 2$ with $0 < |X| \leq \nu$.

$\text{cov}(\mathbf{M}_{\nu,\rho})$ is the least cardinality of any $Y \subseteq \mathbf{M}_{\nu,\rho}$ such that ${}^\rho 2 = \cup Y$.

PROPOSITION 3.1.

- (i) ([L2],[Mil2]) $\text{cov}(\mathbf{M}_{\nu,\rho}) \geq \nu^+$ for every cardinal $\rho \geq \nu$.
- (ii) ([L2],[Mil2]) Suppose that ρ and μ are two cardinals such that $\nu \leq \mu \leq \rho$. Then $\text{cov}(\mathbf{M}_{\nu,\mu}) \geq \text{cov}(\mathbf{M}_{\nu,\rho})$.
- (iii) ([L2]) Suppose $2^{<\nu} > \nu$. Then $\text{cov}(\mathbf{M}_{\nu,\nu}) = \nu^+$.

PROPOSITION 3.2. Suppose that ρ is a cardinal $> \nu$ and $V \models 2^{<\nu} = \nu$. Then setting $P = Fn(\rho, 2, \nu)$:

- (i) ([L2],[Mil2]) $V^P \models \text{cov}(\mathbf{M}_{\nu,\rho}) \geq \rho$.
- (ii) ([L2],[Mil2]) If $cf(\rho) \leq \nu$, then $V^P \models \text{cov}(\mathbf{M}_{\nu,\nu}) > \rho$.
- (iii) Let μ be any regular cardinal $> \nu$. Then $(\mathfrak{d}_\mu)^{V^P} = (\mathfrak{d}_\mu)^V$ and $(\bar{\mathfrak{d}}_\mu)^{V^P} \leq (\bar{\mathfrak{d}}_\mu)^V$.

Proof. (iii) : The conclusion easily follows from the following observation : Suppose that σ is a cardinal > 0 and $F \in V^P$ is a function from $\sigma \times \mu$ to μ . Then by Lemma VII.6.8 of [K], there is $H : \sigma \times \mu \rightarrow P_{\nu^+}(\mu)$ such that $H \in V$ and $F(\alpha, \beta) \in H(\alpha, \beta)$ (so $F(\alpha, \beta) \leq \cup H(\alpha, \beta)$) for every $(\alpha, \beta) \in \sigma \times \mu$. \square

Remark. It is not known whether it is consistent that $cf(\text{cov}(\mathbf{M}_{\nu,\nu})) \leq \nu$.

4. Unequality

Our main concern in this section is with the problem of evaluating the inequality number \mathfrak{U}_κ when κ is a successor cardinal.

Definition. \mathfrak{U}_κ (respectively, \mathfrak{U}'_κ) is the least cardinality of any $F \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in F$ such that $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$ (respectively $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa$).

The following is readily checked.

PROPOSITION 4.1. $\text{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{U}_\kappa \leq \mathfrak{d}_\kappa$.

Remark. It is shown in [MRoS] that if $V \models \text{GCH}$, then there is a κ -complete κ^+ -cc forcing notion P such that

$$V^P \models \text{"}\bar{\mathfrak{d}}_\kappa = \kappa^{+\omega} \text{ and } \text{cov}(\mathbf{M}_{\kappa,\kappa}) = 2^\kappa = \kappa^{+(\omega+1)}\text{"}.$$

For models where $\mathfrak{d}_\kappa > \kappa^+$ see also [CS].

PROPOSITION 4.2. $\mathfrak{U}_\kappa = \mathfrak{U}'_\kappa$.

Proof. Fix $F \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in F$ such that

$$|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa.$$

For $f \in F$ and $\gamma, \delta < \kappa$, define $f_{\gamma, \delta} \in {}^\kappa\kappa$ by : $f_{\gamma, \delta}(\alpha) = f(\alpha)$ if $\alpha \geq \gamma$, and $f_{\gamma, \delta}(\alpha) = \delta$ otherwise. Then for every $g \in {}^\kappa\kappa$, there are $f \in F$ and $\gamma, \delta < \kappa$ such that $\{\alpha \in \kappa : f_{\gamma, \delta}(\alpha) = g(\alpha)\} = \phi$. \square

The following is due to Landver [L2].

PROPOSITION 4.3. $cf(\mathfrak{U}_\kappa) > \kappa$.

Proof. Suppose otherwise. Set $\nu = cf(\mathfrak{U}_\kappa)$ and fix $F \subseteq {}^\kappa\kappa$ so that $|F| = \mathfrak{U}_\kappa$ and for every $g \in {}^\kappa\kappa$, there exists $f \in F$ with $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$. Let $\langle F_\beta : \beta < \nu \rangle$ be such that (a) $|F_\beta| < \mathfrak{U}_\kappa$ for any β , and (b) $\bigcup_{\beta < \nu} F_\beta = F$. Select $A_\beta \subseteq \kappa$ for $\beta < \nu$ so that (i) $|A_\beta| = \kappa$ for every $\beta < \nu$, (ii) $A_\beta \cap A_\gamma = \phi$ whenever $\gamma < \beta < \nu$, and (iii) $\bigcup_{\beta < \nu} A_\beta = \kappa$. For each $\beta < \nu$, there is $g_\beta : A_\beta \rightarrow \kappa$ such that

$$\{\alpha \in A_\beta : (f \upharpoonright A_\beta)(\alpha) = g_\beta(\alpha)\} \neq \phi$$

for every $f \in F_\beta$. Set $g = \bigcup_{\beta < \nu} g_\beta$. Then clearly, $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \neq \phi$ for all $f \in F$. This is a contradiction. \square

We now turn our attention to the task of computing \mathfrak{U}_κ . We begin with the case when κ is a successor cardinal.

THEOREM 4.4. Suppose κ is the successor of a regular infinite cardinal ν . Then

$$\mathfrak{U}_\kappa \geq \min(\mathfrak{d}_\kappa, \text{cov}(\mathbf{M}_{\nu,\kappa})).$$

Proof. Fix $F \subseteq {}^\kappa\kappa$ with $0 < |F| < \min(\mathfrak{d}_\kappa, \text{cov}(\mathbf{M}_{\nu,\kappa}))$. Pick $k : \kappa \rightarrow \kappa - \nu$ so that

$$|\{\alpha < \kappa : k(\alpha) > f(\alpha)\}| = \kappa$$

for every $f \in F$. Select a bijection $j : \kappa \times \nu \rightarrow \kappa$ and a bijection $i_\alpha : k(\alpha) \rightarrow \nu$ for each $\alpha < \kappa$. Given $A \subseteq \kappa$ and $t \in {}^A 2$, define a partial function \bar{t} from κ to κ by stipulating that $\bar{t}(\alpha) = \gamma$ if and only if (a) $\gamma < k(\alpha)$, (b) $\{j(\alpha, \eta) : \eta < i_\alpha(\gamma)\} \subseteq t^{-1}(\{0\})$, and (c) $j(\alpha, i_\alpha(\gamma)) \in t^{-1}(\{1\})$. For $f \in F$, let D_f be the set of all $s \in Fn(\kappa, 2, \nu)$ such that there is $\alpha \in dom(\bar{s})$ with $k(\alpha) > f(\alpha)$ and $\bar{s}(\alpha) = f(\alpha)$. Clearly, each D_f is a dense subset of $Fn(\kappa, 2, \nu)$, so we can find $g \in {}^\kappa 2$ with the property that for every $f \in F$, there is $a \in P_\nu(\kappa)$ with $g \upharpoonright a \in D_f$. Then

$$\{\alpha \in dom(\bar{g}) : \bar{g}(\alpha) = f(\alpha)\} \neq \emptyset$$

for every $f \in F$. □

THEOREM 4.5. Suppose κ is a successor cardinal. Then $\mathfrak{U}_\kappa \geq \bar{\mathfrak{d}}_\kappa$.

Proof. Fix $F \subseteq {}^\kappa\kappa$ with $0 < |F| < \bar{\mathfrak{d}}_\kappa$. Set $\kappa = \nu^+$. Pick $k : \kappa \rightarrow \kappa - \nu$ so that

$$|\{\alpha < \kappa : f(\alpha) < k(\alpha)\}| = \kappa$$

for every $f \in F$. For $\alpha < \kappa$, select a bijection $\pi_\alpha : k(\alpha) \rightarrow \nu$. Given $f \in F$, there exists $i_f \in \nu$ such that the set

$$A_f = \{\alpha < \kappa : f(\alpha) < k(\alpha) \text{ and } \pi_\alpha(f(\alpha)) = i_f\}$$

has size κ . Define $g_f \in {}^\kappa\kappa$ by

$$g_f(\beta) = \text{least } \alpha \in A_f \text{ such that } \alpha \geq \beta.$$

It is shown in [MROs] that $\bar{\mathfrak{d}}_\kappa$ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $h \in {}^\kappa\kappa$, there is $x \in P_\kappa(X)$ such that the set $\{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} g_f(\beta)\}$ is nonstationary in κ . Hence there is $h \in {}^\kappa\kappa$ such that the set

$$B_x = \{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} g_f(\beta)\}$$

is stationary in κ for every $x \in P_\kappa(F)$.

Define $J \subseteq P(\kappa)$ by : $D \in J$ if and only if there is $x \in P_\kappa(F)$ such that $D \cap B_x \in NS_\kappa$. Then J is an ideal on κ . Since $\text{sat}(J) > \nu$ by a result of Ulam (see [Ka], 16.3), there exist pairwise disjoint $D_i \in J^+$ for $i < \nu$ with $\bigcup_{i < \nu} D_i = \kappa$.

Let C be the set of all infinite limit ordinals $\delta < \kappa$ such that $h(\xi) < \delta$ for every $\xi < \delta$. Then C is a closed unbounded subset of κ . Define $t \in {}^\kappa\kappa$ so that for every $\eta < \kappa$, $t(\eta) < k(\eta)$ and $c_\eta \in D_{\pi_\eta(t(\eta))}$, where $c_\eta = \bigcup(C \cap \eta)$.

Now fix $f \in F$. Pick $\zeta \in D_{i_f} \cap C \cap B_{\{f\}}$ and set $\eta = g_f(\zeta)$. Notice that $\zeta \leq \eta$ by the definition of g_f . Also, $\eta \leq h(\zeta)$ since $\zeta \in B_{\{f\}}$. Hence $c_\eta = \zeta$ by the definition of C and the fact that $\zeta \in C$. It now follows from the definition of t and the fact that $\zeta \in D_{i_f}$ that $\pi_\eta(t(\eta)) = i_f$. On the other hand, $\eta \in A_f$ since $\eta = g_f(\zeta)$, so $f(\eta) < k(\eta)$ and $\pi_\eta(f(\eta)) = i_f$. Thus $t(\eta) = f(\eta)$. \square

Remark. It follows from Proposition 4.1 and Theorem 4.5 that $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$ if κ is a successor cardinal and $\mathfrak{d}_\kappa < \kappa^{+\omega}$.

THEOREM 4.6. Suppose that κ is a successor cardinal and $2^{<\kappa} = \kappa$. Then $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$.

Proof. By Proposition 4.1 it suffices to prove that $\mathfrak{U}_\kappa \geq \mathfrak{d}_\kappa$. Set $\kappa = \nu^+$ and select a one-to-one

$$j : \bigcup_{\alpha < \kappa} {}^{[\alpha, \alpha + \nu)} \kappa \longrightarrow \kappa.$$

Now fix $F \subseteq {}^\kappa \kappa$ with $0 < |F| < \mathfrak{d}_\kappa$. Select $g \in {}^\kappa \kappa$ so that for every $f \in F$, there is $\beta_f < \kappa$ with

$$j(f \upharpoonright [\beta_f, \beta_f + \nu)) < g(\beta_f).$$

Let C be the set of all $\gamma < \kappa$ such that $\beta + \nu < \gamma$ and $g(\beta) < \gamma$ for every $\beta < \gamma$. Then C is a closed unbounded subset of κ . Let $< \gamma_\delta : \delta < \kappa >$ be the increasing enumeration of C . For $\delta < \kappa$, set

$$W_\delta = \left\{ t \in \bigcup_{\gamma_\delta \leq \alpha < \gamma_{\delta+1}} {}^{[\alpha, \alpha + \nu)} \kappa : j(t) < \gamma_{\delta+1} \right\}.$$

Then define $k_\delta \in {}^{[\gamma_\delta, \gamma_{\delta+1})} \kappa$ so that for every $t \in W_\delta$, there is $\zeta \in \text{dom}(t)$ with $k_\delta(\zeta) = t(\zeta)$. Set $k = \bigcup_{\delta < \kappa} k_\delta$.

Given $f \in F$, let $\delta_f < \kappa$ be such that $\gamma_{\delta_f} \leq \beta_f < \gamma_{\delta_f+1}$. Then $f \upharpoonright [\beta_f, \beta_f + \nu) \in W_{\delta_f}$. Hence $k(\zeta) = f(\zeta)$ for some $\zeta \in [\beta_f, \beta_f + \nu)$. \square

QUESTION. Is it consistent that κ is a successor cardinal and $\mathfrak{U}_\kappa < \mathfrak{d}_\kappa$?

QUESTION. Is it consistent that κ is a successor cardinal such that $2^{<\kappa} = \kappa$ and $\text{cov}(\mathbf{M}_{\kappa, \kappa}) < \mathfrak{U}_\kappa$?

Let us now consider the case when κ is a limit cardinal. By a result of Bartoszyński [B] and Miller [Mil1], $\mathfrak{U}_\omega = \text{cov}(\mathbf{M}_{\omega, \omega})$. Landver [L2] was able to show that this fact generalizes to uncountable inaccessible cardinals :

THEOREM 4.7. If κ is an inaccessible cardinal, then $\mathfrak{U}_\kappa = \text{cov}(\mathbf{M}_{\kappa, \kappa})$.

QUESTION. Is it consistent that κ is a limit cardinal and $\text{cov}(\mathbf{M}_{\kappa, \kappa}) < \mathfrak{U}_\kappa$?

5. Weak selectivity

The following is due to Baumgartner, Taylor and Wagon [BauTW].

PROPOSITION 5.1. If κ is a successor cardinal, then every ideal on κ is a weak Q -point.

By Proposition 5.1 and Theorem 2.2 $\mathbf{non}_\kappa(\text{weakly selective}) = \mathfrak{d}_\kappa$ if κ is a successor cardinal. The remainder of the section is primarily concerned with the value of $\mathbf{non}_\kappa(\text{weakly selective})$ in the case when κ is a limit cardinal.

Remark. It is easy to see that $\kappa^+ \leq \mathbf{non}_\kappa(\text{weak } Q\text{-point})$ if κ is a limit cardinal.

Definition. An ideal J on κ is a weak semi- Q -point if given $A \in J^+$ and a $< \kappa$ -to-one function f from A to κ , there is $C \in J^+ \cap P(A)$ such that $|C \cap f^{-1}(\{\alpha\})| \leq |\alpha|$ for every $\alpha \in \kappa$.

J is weakly semiselective if J is both a weak semi- Q -point and a weak P -point.

J is weakly rapid if given $A \in J^+$ and $f \in {}^\kappa\kappa$, there is $C \in J^+ \cap P(A)$ such that $\text{o.t.}(C \cap f(\alpha)) \leq \alpha + 1$ for every $\alpha \in \kappa$.

Remark. It is simple to see that every weak Q -point ideal on κ is weakly rapid, and every weakly rapid ideal on κ is a weak semi- Q -point.

Every weak semi- Q -point ideal on ω is weakly rapid ([MP1]). We will show that this does not generalize.

Definition. An ideal J on κ is a semi- Q -point if given a $< \kappa$ -to-one function f from κ to κ , there is $B \in J$ such that $|f^{-1}(\{\alpha\}) - B| \leq |\alpha|$ for every $\alpha \in \kappa$.

PROPOSITION 5.2. Suppose κ is a limit cardinal. Then there exists a semi- Q -point ideal on κ that is not weakly rapid.

Proof. Let Y be the set of all infinite cardinals $< \kappa$. Select $h \in {}^Y\kappa$ so that (a) $h(\mu)$ is a regular infinite cardinal $\leq \mu$ for every $\mu \in Y$, and (b) $\{\mu \in Y : h(\mu) \geq \theta\}$ is stationary in κ for every $\theta \in Y$. For $A \subseteq \kappa$ and $\theta \in Y$, let T_θ^A be the set of all $\mu \in Y$ such that $h(\mu) \geq \theta$ and $|A \cap [\mu, \mu + h(\mu))| = h(\mu)$. Now let J_h be the set of all $A \subseteq \kappa$ such that T_θ^A is a nonstationary subset of κ for some $\theta \in Y$. It is simple to check that J_h is an ideal on κ .

Let us remark in passing that if κ is weakly Mahlo and h is defined by : $h(\mu) = \omega$ if μ is singular, and $h(\mu) = \mu$ otherwise, then a subset A of κ lies in J_h if and only if the set of all $\mu \in Y$ such that μ is regular and $|A \cap [\mu, \mu + \mu)| = \mu$ is nonstationary in κ .

Let us show that J_h is a semi- Q -point. Thus fix a $< \kappa$ -to-one function $f : \kappa \rightarrow \kappa$. Then

$$C = \left\{ \mu \in Y : \mu = \bigcup_{\alpha < \mu} f^{-1}(\{\alpha\}) \right\}$$

is a closed unbounded subset of κ . Set $Q = \bigcup_{\mu \in C} [\mu, \mu + h(\mu))$. It is immediate that $\kappa - Q \in J_h$. Now fix $\alpha \in \kappa$ such that $Q \cap f^{-1}(\{\alpha\}) \neq \emptyset$. Pick $\nu \in C$ so that

$$[\nu, \nu + h(\nu)) \cap f^{-1}(\{\alpha\}) \neq \emptyset.$$

Clearly, $\alpha \geq \nu$ and $\nu \cap f^{-1}(\{\alpha\}) = \emptyset$. Let ρ be the least element of C that is $> \nu$. Then $\alpha < \rho$ and $f^{-1}(\{\alpha\}) \subseteq \rho$. Thus

$$Q \cap f^{-1}(\{\alpha\}) \subseteq [\nu, \nu + h(\nu))$$

and consequently

$$|Q \cap f^{-1}(\{\alpha\})| \leq h(\nu) \leq \nu \leq |\alpha|.$$

It remains to show that J is not weakly rapid. Fix $D \in J_h^+$. Then

$$S = \{\mu \in T_\omega^D : |T_\omega^D \cap \mu| = \mu\}$$

is a stationary subset of κ . Given $\mu \in S$, $|D \cap \mu| = \mu$ since

$$D \cap [\rho, \rho + h(\rho)) \subset D \cap \mu$$

for every $\rho \in \mu \cap T_\omega^D$, and hence

$$\text{o.t.}(D \cap (\mu + h(\mu))) > \mu + 1.$$

□

THEOREM 5.3. Suppose κ is a limit cardinal. Then $\mathfrak{U}_\kappa \leq \text{non}_\kappa(\text{weak semi-}Q\text{-point})$.

Proof. Let J be an ideal on κ with $cof(J) < \mathfrak{U}_\kappa$. Let us show that J is a weak semi- Q -point. Thus fix $A \in J^+$ and a $< \kappa$ -to-one function $f : A \rightarrow \kappa$. Select $B_\beta \in J$ for $\beta < cof(J)$ so that $J = \bigcup_{\beta < cof(J)} P(B_\beta)$.

For $\beta < cof(J)$, define $g_\beta \in {}^\kappa\kappa$ by :

$$g_\beta(\alpha) = \text{least element of } (\bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\})) - B_\beta.$$

There is $h \in {}^\kappa\kappa$ such that $\{\alpha \in \kappa : g_\beta(\alpha) = h(\alpha)\} \neq \emptyset$ for every $\beta < cof(J)$. Define $C \subseteq ran(h)$ by : $h(\alpha) \in C$ just in case $h(\alpha) \in \bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\})$. Then clearly $C \in J^+ \cap P(A)$. Moreover, $C \cap f^{-1}(\{\alpha\}) \subseteq \{h(\gamma) : \gamma < \alpha\}$ for every $\alpha < \kappa$. \square

THEOREM 5.4. Suppose that κ is a limit cardinal and $2^{<\kappa} = \kappa$. Then

$$\overline{\text{non}}_\kappa(\text{weakly semiselective}) \leq \mathfrak{U}_\kappa \leq \text{non}_\kappa(\text{weak } Q\text{-point}).$$

Proof. The proof of the first inequality is an easy modification of that of Lemma 6.1 in [MP1] (which should be corrected by substituting “ $e \in [\omega]^{<\omega}$ such that $B \subseteq \bigcup_{j \in e} \omega^{E_j} \cup \bigcup_{f \in z} B_f$ ” for “ $e \in [\bigcup_{j \in \omega} \omega^{E_j}]^{<\omega}$ such that $B \subseteq e \cup \bigcup_{f \in z} B_f$ ”). The second inequality is proved as Proposition 5.3 in [MP1]. \square

Remark. Suppose that κ is a limit cardinal, $2^{<\kappa} = \kappa$ and $\text{non}_\kappa(\text{weakly semiselective}) < \kappa^{+\omega}$. Then by Proposition 4.1 and Theorems 2.2 and 5.4,

$$\mathfrak{U}_\kappa = \text{non}_\kappa(\text{weakly selective}) = \text{non}_\kappa(\text{weakly semiselective}).$$

Remark. It is consistent (see [MP1]) that $\mathfrak{U}_\omega < \text{non}_\omega(\text{weak } Q\text{-point})$, and that $\text{non}_\omega(\text{weak } Q\text{-point}) < \text{non}_\omega(\text{weak semi-}Q\text{-point})$. We do not know whether these results can be generalized.

QUESTION. Is it consistent that κ is a limit cardinal, $2^{<\kappa} > \kappa$ and $\kappa^+ < \text{non}_\kappa(\text{weak } Q\text{-point})$?

QUESTION. By a result of [MP1], $cf(\text{non}_\omega(\text{weak } Q\text{-point})) > \omega$. Does this generalize ?

6. $\text{non}_\kappa(J^+ \rightarrow (J^+, \theta)^2)$

In this section we use standard material to discuss the value of $\text{non}_\kappa(J^+ \rightarrow (J^+, \theta)^2)$ for a cardinal $\theta \in [3, \kappa]$.

THEOREM 6.1.

- (i) $\mathfrak{d}_\kappa \geq \text{non}_\kappa(J^+ \rightarrow (J^+, 3)^2)$.
- (ii) $\overline{\mathfrak{d}}_\kappa \geq \overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, 3)^2)$.
- (iii) $\overline{\text{non}}_\kappa(\text{weak } P\text{-point}) \geq \overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, \omega)^2)$.

Proof. (i) and (ii) : By a straightforward generalization of Lemma 4.4 in [M2], there exists an ideal J on κ such that $cof(J) \leq \overline{\mathfrak{d}}_\kappa$, $cof(J) \leq \mathfrak{d}_\kappa$ and $J^+ \not\rightarrow (J^+, 3)^2$.

(iii) : Baumgartner, Taylor and Wagon [BauTW] established that if J is an ideal on κ such that $J^+ \rightarrow (J^+, \omega)^2$, then J is a weak P -point. \square

Definition. Given an ideal J on κ , $A \in J^+$ and $F : \kappa \times \kappa \rightarrow 2$, (J, A, F) is 0-good if there is $D \in J^+ \cap P(A)$ such that $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in D$.

The following is readily checked.

LEMMA 6.2. Suppose that J is weakly selective and (J, A, F) is 0-good, where J is an ideal on κ , $A \in J^+$ and $F : \kappa \times \kappa \rightarrow 2$. Then there is $B \in J^+ \cap P(A)$ such that F is constantly 0 on $[B]^2$.

LEMMA 6.3. Suppose that (J, A, F) is not 0-good, where J is an ideal on κ , $A \in J^+$ and $F : \kappa \times \kappa \rightarrow 2$. Then :

- (i) There is $B \subseteq A$ such that $\text{o.t.}(B) = \omega + 1$ and F is identically 1 on $[B]^2$.
- (ii) Suppose that $\mathfrak{a}_J > \kappa$ and θ is an uncountable cardinal $< \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$. Then there is $C \subseteq A$ such that $\text{o.t.}(C) = \theta + 1$ and F is identically 1 on $[C]^2$.

Proof. The proof is similar to that of Lemma 10.4 below. □

THEOREM 6.4.

- (i) $\overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, \omega + 1)^2) \geq \overline{\text{non}}_\kappa(\text{weakly selective})$.
- (ii) Suppose that θ is an infinite cardinal $< \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$. Then

$$\text{non}_\kappa(J^+ \rightarrow (J^+, \theta + 1)^2) \geq \text{non}_\kappa(\text{weakly selective}).$$

Proof. (i) : Baumgartner, Taylor and Wagon [BauTW] showed that $J^+ \rightarrow (J^+, \omega + 1)^2$ for every weakly selective ideal J on κ .

(ii) : By Lemmas 6.2 and 6.3. □

Remark. Suppose that κ is a successor cardinal and θ is cardinal ≥ 2 such that $\kappa \rightarrow (\kappa, \theta)^2$. Then by Theorems 6.1 (i), 6.4 (ii) and 2.2 and Proposition 5.1, $\mathfrak{d}_\kappa = \text{non}_\kappa(J^+ \rightarrow (J^+, \theta + 1)^2)$.

Remark. It is consistent (see [M2]) that $\mathfrak{d} > \text{non}_\omega(J^+ \rightarrow (J^+, 3)^2)$. We do not know whether this can be generalized.

THEOREM 6.5. Suppose κ is a weakly compact cardinal. Then :

- (i) $\overline{\text{non}}_\kappa(\text{weak } Q\text{-point}) \geq \overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, \kappa)^2)$.
- (ii) $\text{non}_\kappa(J^+ \rightarrow (J^+)^2) = \text{non}_\kappa(J^+ \rightarrow (J^+, \kappa)^2) = \text{non}_\kappa(\text{weakly selective})$.

Proof. The result follows from Theorems 2.2 and 6.1 (i) and the following two well-known facts : (1) Every ideal J on κ such that $J^+ \rightarrow (J^+, \kappa)^2$ is a weak Q -point ; (2) If κ is weakly compact, then $J^+ \rightarrow (J^+)^2$ for every weakly selective ideal J on κ such that $\mathfrak{a}_J > \kappa$. □

7. $\text{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$

In this section we consider the cardinal $\text{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$, where $3 \leq \rho \leq \kappa$, about which little is known. We begin with the case where $\rho = 3$. The following is due to Blass [Bl].

LEMMA 7.1. Suppose J is an ideal on κ such that $J^+ \rightarrow [J^+]_3^2$. Then J is a weak P-point.

Proof. Fix $A \in J^+$ and $f \in {}^A\kappa$ with $\{f^{-1}(\{\gamma\}) : \gamma \in \kappa\} \subseteq J$. Define $g : [A]^2 \rightarrow 3$ by stipulating that $g(\alpha, \beta) = 0$ if and only if $f(\alpha) < f(\beta)$, and $g(\alpha, \beta) = 1$ if and only if $f(\alpha) = f(\beta)$. There are $B \in J^+ \cap P(A)$ and $i < 3$ such that $i \notin g''[B]^2$. It is simple to see that $i \neq 0$, so f is $< \kappa$ -to-one on B . \square

The following is proved by adapting an argument of Baumgartner and Taylor [BauT].

LEMMA 7.2. Suppose J is an ideal on κ such that $J^+ \rightarrow [J^+]_3^2$, and (J, A, F) is 0-good, where $A \in J^+$ and $F : \kappa \times \kappa \rightarrow 2$. Then either there exists $C \in J^+ \cap P(A)$ such that F is constantly 0 on $[C]^2$, or for every $\delta < \kappa$, there exists $Q \subseteq A$ such that $\text{o.t.}(Q) = \delta$ and F is constantly 1 on $[Q]^2$.

Proof. Select $B \in J^+ \cap P(A)$ so that $\{\beta \in B : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in B$. By Lemma 7.1, there exists $S \in J^+ \cap P(B)$ so that $|\{\beta \in S : F(\alpha, \beta) = 1\}| < \kappa$ for every $\alpha \in S$. Define δ_ξ for $\xi < \kappa$ by :

- (i) $\delta_0 = \cap S$;
- (ii) $\delta_{\xi+1}$ = the least $\zeta < \kappa$ with the property that $\zeta > \beta$ for every $\beta \in S$ such that $F(\alpha, \beta) = 1$ for some $\alpha \in S \cap \delta_\zeta$;
- (iii) $\delta_\xi = \bigcup_{\zeta < \xi} \delta_\zeta$ if ξ is a limit ordinal > 0 .

Let X be the set of all limit ordinals $< \kappa$. For $\eta \in X$, $n \in \omega$ and $j < 2$, set

$$d_{\eta, n}^j = S \cap [\delta_{\eta+2n+j}, \delta_{\eta+2n+j+1}).$$

For $j < 2$, let

$$D^j = \bigcup \{d_{\eta, n}^j : \eta \in X \text{ and } n \in \omega\}.$$

Select $k < 2$ so that $D^k \in J^+$. Notice that $F(\alpha, \beta) = 0$ if $(\alpha, \beta) \in [D^k]^2$ and $\{\alpha, \beta\} \not\subseteq d_{\eta, n}^k$ for all $\eta \in X$ and $n \in \omega$.

Define $h : [D^k]^2 \rightarrow 3$ by stipulating that $h(\alpha, \beta) = 0$ if and only if $\{\alpha, \beta\} \not\subseteq d_{\eta, n}^k$ for all $\eta \in X$ and $n \in \omega$, and $h(\alpha, \beta) = 1$ if and only if $F(\alpha, \beta) = 1$. There are $W \in J^+ \cap P(D^k)$ and $i < 3$ so that $i \notin h''[W]^2$. Clearly, $i \neq 0$. If $i = 1$, F is identically 0 on $[W]^2$. Now assume $i = 2$. Let Z be the set of all $(\eta, n) \in X \times \omega$ such that $W \cap d_{\eta, n}^k \neq \emptyset$. Suppose that there is $\gamma < \kappa$ such that $\text{o.t.}(W \cap d_{\eta, n}^k) \leq \gamma$ for every $(\eta, n) \in Z$. Then there exists $C \in J^+ \cap P(W)$ such that $|C \cap d_{\eta, n}| = 1$ for any $(\eta, n) \in Z$. Clearly, F takes the constant value 0 on $[T]^2$. \square

PROPOSITION 7.3. Suppose $\theta \in (2, \kappa)$ is a cardinal such that $\kappa \rightarrow (\kappa, \theta)^2$. Then

$$\text{non}_\kappa(J^+ \rightarrow [J^+]_3^2) \leq \text{non}_\kappa(J^+ \rightarrow (J^+, \theta + 1)^2).$$

Proof. By Theorem 2.2 and Lemmas 6.3, 7.1 and 7.2. \square

Let us now consider the partition relation $J^+ \rightarrow [J^+]_\kappa^2$. We begin with the following observation.

PROPOSITION 7.4. Suppose κ is inaccessible. Then there is an ideal J on κ such that (a) $J^+ \not\rightarrow [J^+]_\kappa^2$, (b) J is not a weak semi-Q-point, (c) $\mathfrak{a}_J > \kappa$, and (d) $J^+ \rightarrow (J^+, \alpha)^2$ for every $\alpha < \kappa$.

Proof. Let $\langle \rho_\alpha : \alpha < \kappa \rangle$ be the increasing enumeration of all strong limit infinite cardinals $< \kappa$. Let Z be the set of all regular infinite cardinals $< \kappa$. For $\mu \in Z$, set $\nu_\mu = (\rho_\mu)^{++}$. Then $\nu_\mu \not\rightarrow [\nu_\mu]_{\nu_\mu}^2$ by a result of Todorcevic [To1]. On the other hand, by a result of Erdős and Rado (see [EHMáR], Corollary 17.5), $\nu_\mu \rightarrow (\nu_\mu, \tau)^2$ for every infinite cardinal $\tau < \mu$. Pick pairwise disjoint A_μ for $\mu \in Z$ so that $|A_\mu| = \nu_\mu$ for any $\mu \in Z$, and $\bigcup_{\mu \in Z} A_\mu = \kappa$. Let J be the set of all $B \subseteq \kappa$ such that

$$|\{\mu \in Z : |B \cap A_\mu| = \nu_\mu\}| < \kappa.$$

It is simple to see that J is an ideal on κ .

For $\mu \in Z$, pick $g_\mu : [A_\mu]^2 \rightarrow \nu_\mu$ so that $g_\mu''[B]^2 = \nu_\mu$ for every $B \subseteq A_\mu$ with $|B| = \nu_\mu$. Let $G : [\kappa]^2 \rightarrow \kappa$ be such that $\bigcup_{\mu \in Z} g_\mu \subseteq G$. Then clearly $G''[C]^2 = \kappa$ for any $C \in J^+$.

Define $f \in {}^\kappa \kappa$ by stipulating that $f^{-1}(\{\mu\}) = A_\mu$ for every $\mu \in Z$. Clearly, there is no $S \in J^+$ so that $|S \cap f^{-1}(\{\alpha\})| \leq |\alpha|$ for all $\alpha < \kappa$. Hence J is not a weak semi-Q-point.

Let us next show that $\mathfrak{a}_J > \kappa$. Thus suppose that $B_\alpha \in J^+$ for $\alpha < \kappa$, and $B_\alpha \cap B_\beta \in J$ whenever $\beta < \alpha < \kappa$. Select a strictly increasing function $k : \kappa \rightarrow Z$ so that

$$|(B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}| = \nu_{k(\alpha)}$$

for any $\alpha < \kappa$. Set

$$T = \bigcup_{\alpha < \kappa} ((B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}).$$

Then $T \in J^+$ and moreover $|T \cap B_\alpha| < \kappa$ for every $\alpha < \kappa$.

It remains to prove (d). Thus fix $A \in J^+$ and $F : \kappa \times \kappa \rightarrow 2$. Suppose that there is $\eta < \kappa$ such that for every $Q \subseteq A$ with $\text{o.t.}(Q) = \eta$, F is not constantly 1 on $[Q]^2$. Since by Theorem 17.1 of [EHMáR] $\kappa \rightarrow (\kappa, \alpha)^2$ for every $\alpha < \kappa$, it follows from Lemma 6.3 that (J, A, F) is 0-good. Select $D \in J^+ \cap P(A)$ so that $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in D$. Define D_γ for $\gamma < \kappa$ and a strictly increasing function $h : \kappa \rightarrow Z$ so that

$$(0) \quad D_\gamma = D - \left(\bigcup_{\delta < \gamma} \bigcup_{\alpha \in D_\delta \cap A_{h(\delta)}} \{\beta \in D : F(\alpha, \beta) = 1\} \right);$$

$$(1) \quad |D_\gamma \cap A_{h(\gamma)}| = \nu_{h(\gamma)}.$$

For $\gamma \in (|\eta|^+, \kappa)$, select $X_\gamma \subseteq D_\gamma \cap A_{h(\gamma)}$ so that $|X_\gamma| = \nu_{h(\gamma)}$ and F is constantly 0 on $[X_\gamma]^2$. Set $Y = \bigcup_{|\eta|^+ < \gamma < \kappa} X_\gamma$. Then clearly $Y \in J^+ \cap P(A)$. Moreover, F takes the constant value 0 on $[Y]^2$. \square

Remark. $J^+ \rightarrow (J^+, \kappa)^2$ does not necessarily imply that $J^+ \rightarrow [J^+]_\kappa^2$. This follows from the following two facts : (0) If κ is weakly compact, then there exists a normal ideal J on κ such that $J^+ \rightarrow (J^+, \kappa)^2$ ([Bau1], [Bau2]) ; (1) Assuming $V = L$, κ is completely ineffable if and only if there is a normal ideal J on κ such that $J^+ \rightarrow [J^+]_\kappa^2$ ([M4]).

Recall that for $S \subseteq \kappa$, $\diamondsuit_\kappa^*(S)$ means that there are $s_\alpha \in P_{|\alpha|^+}(\alpha)$ for $\alpha \in S$ such that for every $A \subseteq \kappa$, there exists a closed unbounded subset C of κ with the property that $A \cap \alpha \in s_\alpha$ for every $\alpha \in C \cap S$.

PROPOSITION 7.5.- Suppose that $\diamondsuit_\kappa^*(S)$ holds for some stationary subset S of κ . Then $\mathfrak{d}_\kappa \geq \text{non}_\kappa(J^+ \rightarrow [J^+]_\kappa^2)$ and $\overline{\mathfrak{d}}_\kappa \geq \overline{\text{non}}_\kappa(J^+ \rightarrow [J^+]_\kappa^2)$.

Proof. By a result of [M4], the hypothesis implies that $NS_\kappa^+ \not\rightarrow [NS_\kappa^+]_\kappa^2$. \square

Remark. It is shown in [S] that if (a) κ is a successor cardinal $\geq \omega_2$ with $2^{<\kappa} = \kappa$, and (b) setting $\kappa = \nu^+$, $\mu^\tau \leq \nu$ for every infinite cardinal $\mu < \nu$, where $\tau = \aleph_1$ if $cf(\nu) = \omega$ and $\tau = \aleph_0$ otherwise, then there is a stationary subset S of κ such that $\diamondsuit_\kappa^*(S)$ holds.

Remark. We do not know whether it is consistent that the conclusion of Proposition 7.5 fails. Results of Section 15 (below) imply that

$$\text{non}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa) \leq (\overline{\mathfrak{d}}_\kappa)^{<\kappa}$$

if κ is a limit cardinal such that $2^{<\kappa} = \kappa$.

8. $\mathfrak{d}_{\kappa,\lambda}^\kappa$

We now start our study of combinatorial properties of ideals on $P_\kappa(\lambda)$. The aim of this section is to present a two-cardinal version of Theorem 2.2.

Definition. $\mathfrak{d}_{\kappa,\lambda}^\kappa$ is the least cardinality of any $F \subseteq {}^\kappa(P_\kappa(\lambda))$ with the property that for every $g \in {}^\kappa(P_\kappa(\lambda))$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for all $\alpha \in \kappa$.

Remark. It is shown in [MPéS1] that $\mathfrak{d}_{\kappa,\lambda}^\kappa = \mathfrak{d}_\kappa \cdot u(\kappa^+, \lambda)$.

Definition. Given an ideal H on $P_\kappa(\lambda)$, $\mathcal{M}_H^{>\kappa}$ is the set of all $Q \subseteq H^+$ such that (i) $|Q| \geq \kappa$, (ii) $A \cap B \in H$ for all $A, B \in Q$ with $A \neq B$, and (iii) for every $C \in H^+$, there is $A \in Q$ with $A \cap C \in H^+$. \mathfrak{a}_H is the least cardinality of any member of $\mathcal{M}_H^{>\kappa}$ if $\mathcal{M}_H^{>\kappa} \neq \emptyset$, and $2^{(\lambda^{<\kappa})^+}$ otherwise.

The following is proved as Proposition 11.2 of [MP2].

PROPOSITION 8.1. Given a κ -normal ideal H on $P_\kappa(\lambda)$, the following are equivalent :

- (i) $\mathfrak{a}_H = \kappa$.
- (ii) $\text{sat}(H) > \kappa$.

COROLLARY 8.2. Let $A \in (NS_{\kappa,\lambda}^\kappa)^+$ and set $H = NS_{\kappa,\lambda}^\kappa \mid A$. Then $\mathfrak{a}_H = \kappa$.

Proof. The result follows from Proposition 8.1 since $\text{sat}(H) > \kappa$ by a result of Abe [A]. \square

The following is proved as Proposition 11.1 (ii) of [MP2].

PROPOSITION 8.3. Given an ideal H on $P_\kappa(\lambda)$, the following are equivalent :

- (i) $\mathfrak{a}_H = \kappa$.
- (ii) There exist $A_\alpha \in H^+$ for $\alpha < \kappa$ such that (a) $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha < \kappa$, and (b) for every $C \in H^+$, there is $\alpha < \kappa$ such that $C - A_\alpha \in H^+$.

Definition. An ideal H on $P_\kappa(\lambda)$ is a weak π -point if given $f \in {}^\kappa H$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $B \cap f(\alpha) \in I_{\kappa,\lambda}$ for every $\alpha \in \kappa$.

THEOREM 8.4. Let H be an ideal on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$. Then $\mathfrak{a}_H > \kappa$ and H is a weak π -point.

Proof. Let $A_\alpha \in H^+$ for $\alpha < \kappa$ be such that $A_\alpha \subseteq A_\beta$ for all $\beta < \alpha$. Select $X \subseteq H$ so that $|X| = \text{cof}(H)$ and $H = \bigcup_{B \in X} P(B)$. For $B \in X$, define $f_B \in {}^\kappa(P_\kappa(\lambda))$ so that $f_B(\alpha) \in A_\alpha - B$. There is $g \in {}^\kappa(P_\kappa(\lambda))$ such

that $\{\alpha < \kappa : g(\alpha) \not\subseteq f_B(\alpha)\} \neq \emptyset$ for every $B \in X$. Set $C = \bigcup_{\alpha < \kappa} \{a \in A_\alpha : g(\alpha) \not\subseteq a\}$. Then $C \in H^+$, and moreover $C - A_\alpha \in I_{\kappa, \lambda}$ for any $\alpha < \kappa$.

Definition. $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ is the least cardinality of any $X \subseteq {}^\kappa(P_\kappa(\lambda))$ with the property that for every $g \in {}^\kappa(P_\kappa(\lambda))$, there is $x \in P_\kappa(X)$ such that $g(\alpha) \subseteq \bigcup_{f \in x} f(\alpha)$ for every $\alpha < \kappa$.

Remark. It is shown in [MRoS] that $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa = \overline{\mathfrak{d}}_\kappa \cdot \text{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$, where $\text{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$ denotes the least cardinality of any $X \subseteq P_{\kappa^+}(\lambda)$ such that for every $b \in P_{\kappa^+}(\lambda)$, there is $x \in P_\kappa(X)$ with $b \subseteq \cup x$.

Remark. It is immediate that $I_{\kappa, \lambda}$ is a weak π -point. On the other hand $\mathfrak{a}_{I_{\kappa, \lambda}} > \kappa$ does not necessarily hold. In fact if $cf(\lambda) \neq \kappa$ and $\overline{\mathfrak{d}}_{\kappa, \sigma}^\kappa \leq \lambda$ for every cardinal $\sigma \in [\kappa, \lambda)$, then $\mathfrak{a}_{I_{\kappa, \lambda}} = \kappa$ ([M6]).

LEMMA 8.5. Suppose that H is an ideal on $P_\kappa(\lambda)$ with $\mathfrak{a}_H = \kappa$. Then there is an ideal K on $P_\kappa(\lambda)$ such that (a) K is not a weak π -point, (b) $cof(K) \leq cof(H)$, and (c) $\overline{cof}(K) \leq \overline{cof}(H)$.

Proof. Select $A_\alpha \in H^+$ for $\alpha < \kappa$ so that (α) $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha < \kappa$, and (β) for any $C \in H^+$, there is $\alpha < \kappa$ with $C - A_\alpha \in H^+$. Let K be the set of all $B \subseteq P_\kappa(\lambda)$ such that $B \cap A_\alpha \in H$ for some $\alpha < \kappa$. It is simple to check that K is as desired. \square

THEOREM 8.6.

- (i) There is an ideal H on $P_\kappa(\lambda)$ such that (a) $\mathfrak{a}_H = \kappa$, (b) $cof(H) = \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$, and (c) $\overline{cof}(H) \leq \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$.
- (ii) There is an ideal K on $P_\kappa(\lambda)$ such that (a) K is not a weak π -point, (b) $cof(K) = \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$, and (c) $\overline{cof}(K) \leq \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$.

Proof. (i) : Set $H = NS_{\kappa, \lambda}^\kappa$. Then $\mathfrak{a}_H = \kappa$ by Corollary 8.2. Moreover, $cof(H) = \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ ([MPéS1]) and $\overline{cof}(H) = \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ ([MRoS]).

(ii) : By (i), Lemma 8.5 and Theorem 8.4. \square

Remark. Theorem 8.6 is not optimal, even under GCH. In fact, suppose that the GCH holds, $\lambda = \sigma^+$, where σ is a cardinal of cofinality $< \kappa$, and κ is not the successor of a cardinal of cofinality $\leq cf(\sigma)$. Then $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa = \lambda$ ([MRoS]). Moreover, there is $A \in (NS_{\kappa, \lambda}^\kappa)^+$ such that $\overline{cof}(NS_{\kappa, \lambda}^\kappa \mid A) = \sigma$ ([MPéS2]). Hence there is by Corollary 8.2 an ideal H on $P_\kappa(\lambda)$ (namely $H = NS_{\kappa, \lambda}^\kappa \mid A$) such that $\overline{cof}(H) < \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ and $\mathfrak{a}_H = \kappa$, and by Lemma 8.5 an ideal K on $P_\kappa(\lambda)$ such that $\overline{cof}(K) < \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ and K is not a weak π -point.

9. Weak χ -pointness

Definition. An ideal H on $P_\kappa(\lambda)$ is a weak χ -point if given $A \in H^+$ and $g \in {}^\kappa(P_\kappa(\lambda))$, there is $B \in H^+ \cap P(A)$ such that $g(\cup(a \cap \kappa)) \subseteq b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

Our primary concern in this section is with the problem of determining when $I_{\kappa, \lambda}$ is a weak χ -point. We will first give a sufficient condition and then prove that this condition is necessary if κ is inaccessible.

The following is proved as Lemma 2.1 in [M2].

THEOREM 9.1. Let H be an ideal on $P_\kappa(\lambda)$ such that $cof(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$. Then H is a weak χ -point.

QUESTION. Is it consistent that $2^{<\kappa} > \kappa$ and I_{κ, κ^+} is a weak χ -point ?

THEOREM 9.2. Suppose that for all $A \in I_{\kappa, \lambda}^+$ with $A \subseteq \{a : \cup(a \cap \kappa) \in a\}$, there is $B \in I_{\kappa, \lambda}^+ \cap P(A)$ such that $\cup(a \cap \kappa) \in b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Then $\sigma < \text{non}_\kappa$ (weakly selective) for every $\sigma \in \mathcal{K}(\kappa, \lambda)$.

Proof. Suppose that $T \subseteq P_\kappa(\lambda - \kappa)$ is such that $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$, and J is an ideal on κ with $\text{cof}(J) \leq |T|$. Select $D_d \in J$ for $d \in T$ so that for every $W \in J$, there is $u \in P_\kappa(T) - \{\phi\}$ with $W \subseteq \bigcup_{d \in u} D_d$. Now fix $G_\alpha \in J$ for $\alpha < \kappa$. Define $A \subseteq P_\kappa(\lambda)$ by stipulating that $a \in A$ if and only if there is $\delta < \kappa$ such that (a) $\delta = \max(a \cap \kappa)$, (b) $\delta \notin \bigcup_{d \in T \cap P(a)} D_d$, and (c) $\delta \notin G_\alpha$ for every $\alpha \in a \cap \delta$.

Let us show that $A \in I_{\kappa, \lambda}^+$. Given $c \in P_\kappa(\lambda)$, pick $\delta < \kappa$ so that $\delta \notin \bigcup_{d \in T \cap P(c)} D_d$ and for every $\alpha \in c \cap \kappa, \delta > \alpha$ and $\delta \notin G_\alpha$. Set $e = c \cup \{\delta\}$. Then $e \in A$.

By our assumption there is $B \in I_{\kappa, \lambda}^+ \cap P(A)$ such that $\cup(a \cap \kappa) \in b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Set $C = \{\cup(a \cap \kappa) : a \in B\}$. Then $C \in J^+$. Moreover, $\xi \notin G_\zeta$ for all $\zeta, \xi \in C$ with $\zeta < \xi$. \square

We mention the following partial converse to Theorem 9.2.

PROPOSITION 9.3. Suppose that $2^{<\kappa} = \kappa$ and H is an ideal on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{non}_\kappa$ (weakly selective). Then for all $f \in {}^\kappa\kappa$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $f(\cup(a \cap \kappa)) \subseteq b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

Proof. Fix $f \in {}^\kappa\kappa$ and $A \in H^+$. For $D \subseteq P_\kappa(\kappa)$, set $Z_D = \{a \in P_\kappa(\lambda) : a \cap \kappa \in D\}$. It is simple to see that (a) $Z_{P_\kappa(\kappa)} = P_\kappa(\lambda)$, (b) $Z_{\bigcup \mathfrak{D}} Z_D$ for $\mathfrak{D} \subseteq P(P_\kappa(\kappa))$, (c) $Z_D \in I_{\kappa, \lambda}$ for every $D \subseteq P_\kappa(\kappa)$ with $|D| = 1$, and (d) $Z_{D'} \subseteq Z_D$ for all $D, D' \subseteq P_\kappa(\kappa)$ such that $D' \subseteq D$. Hence

$$K = \{D \subseteq P_\kappa(\kappa) : Z_D \in H \mid A\}$$

is a κ -complete ideal on $P_\kappa(\kappa)$. For $C \subseteq P_\kappa(\lambda)$, let W_C be the set of all $d \in P_\kappa(\kappa)$ such that

$$\{a \in P_\kappa(\lambda) : a \cap \kappa = d\} \subseteq C.$$

If $C \in H \mid A$, then $W_C \in K$ since $Z_{W_C} \subseteq C$. Moreover, if $D \subseteq P_\kappa(\kappa)$ and $Z_D \subseteq C \subseteq P_\kappa(\lambda)$, then $D \subseteq W_C$. Hence

$$\text{cof}(K) \leq \text{cof}(H \mid A) \leq \text{cof}(H).$$

For $d \in P_\kappa(\kappa)$, let S_d be the set of all $e \in P_\kappa(\kappa)$ such that $f(\cup d) \not\subseteq e$ or $\cup e \leq \cup d$. Then $S_d \in K$ since

$$\{a \in Z_{S_d} : f(\cup d) \cup \{(\cup d) + 1\} \subseteq a\} = \phi.$$

Select a bijection $\ell : P_\kappa(\kappa) \rightarrow \kappa$. Since $\text{cof}(K) < \text{non}_\kappa$ (weakly selective), there is $D \in K^+$ such that $e \notin S_d$ for all $d, e \in D$ such that $\ell(d) < \ell(e)$. Set

$$B = A \cap Z_D = \{a \in A : a \cap \kappa \in D\}.$$

Then $B \in H^+$. Now fix $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Then clearly $\ell(a \cap \kappa) \neq \ell(b \cap \kappa)$. In fact $\ell(a \cap \kappa) < \ell(b \cap \kappa)$ (since otherwise $a \cap \kappa \notin S_{b \cap \kappa}$ and therefore $\cup(a \cap \kappa) > \cup(b \cap \kappa)$). Hence $b \cap \kappa \notin S_{a \cap \kappa}$, so $f(\cup(a \cap \kappa)) \subseteq b \cap \kappa$. \square

Definition. For $A \subseteq P_\kappa(\lambda)$, let

$$[A]_\kappa^2 = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}$$

Remark.

$$[P_\kappa(\lambda)]_\kappa^2 = \{(\alpha, b) \in \kappa \times P_\kappa(\lambda) : \alpha < \cup(b \cap \kappa)\}.$$

Definition. For $a, b \in P_\kappa(\lambda)$, let $a \prec b$ just in case $a \subseteq b$ and $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

Definition. For $A \subseteq P_\kappa(\lambda)$, let

$$[A]_\prec^2 = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } a \prec b\}.$$

Remark. $[P_\kappa(\lambda)]_\prec^2 = [P_\kappa(\lambda)]_\kappa^2$.

THEOREM 9.4. Suppose that κ is inaccessible and H is an ideal on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$, and let $A \in H^+$. Then there is $C \in H^+ \cap P(A)$ such that $[C]_\kappa^2 = [C]_\prec^2$.

Proof. For $\alpha < \kappa$, set $A_\alpha = \{a \in A : \cup(a \cap \kappa) = \alpha\}$. By induction on $\alpha < \kappa$, we define $c_k \in \{\phi\} \cup A_\alpha$ for $k \in {}^\alpha 2$ as follows. Given $k \in {}^\alpha 2$, set

$$e_k = \bigcup \{c_{k \upharpoonright \beta} : \beta \in k^{-1}(\{1\})\}$$

and

$$Z_k = \{a \in A_\alpha : e_k \subseteq a\}.$$

If $Z_k \neq \phi$, let c_k be an arbitrary member of Z_k . Otherwise let $c_k = \phi$.

Set $\nu = \text{cof}(H)$ and pick $B_\xi \in H$ for $\xi < \nu$ so that $H = \bigcup_{\xi < \nu} P(B_\xi)$. Let $\xi < \nu$. For $\alpha < \kappa$, let D_ξ^α

be the set of all $s \in {}^{(\alpha+1)} 2$ such that (i) $s(\alpha) = 1$, and (ii) there is $a \in A_\alpha - B_\xi$ with the property that

$$(\forall \beta \in \alpha \cap s^{-1}(\{1\})) (\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Then let $D_\xi = \bigcup_{\alpha < \kappa} D_\xi^\alpha$ and $U_\xi = \bigcup_{s \in D_\xi} O_s^\kappa$. Let us prove that the open set U_ξ is dense. Thus let $\gamma < \kappa$

and $p \in {}^\gamma 2$. Pick $a \in (\bigcup_{\gamma \leq \delta < \kappa} A_\delta) - B_\xi$ so that

$$(\forall \beta \in p^{-1}(\{1\})) (\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Set $\alpha = \cup(a \cap \kappa)$ and define $s \in {}^{(\alpha+1)} 2$ by : $s \restriction \gamma = p$, $s(\delta) = 0$ if $\gamma \leq \delta < \alpha$, and $s(\alpha) = 1$. It is immediate that $s \in D_\xi^\alpha$.

Select $f \in \bigcap_{\xi < \nu} U_\xi$. For each $\xi < \nu$, there is $s_\xi \in D_\xi$ such that $s_\xi \subseteq f$. Let $\alpha_\xi < \kappa$ be such that $s_\xi \in D_\xi^{\alpha_\xi}$. Set $T = \{\alpha_\xi : \xi < \nu\}$ and define $g \in {}^\kappa 2$ so that $g^{-1}(\{1\}) = T$. For $\xi < \nu$, set

$$d_\xi = \bigcup \{c_{g \upharpoonright \beta} : \beta \in T \cap \alpha_\xi\}$$

and

$$C_\xi = \{b \in A_{\alpha_\xi} : d_\xi \subseteq b\}.$$

Finally, let $C = \bigcup_{\xi < \nu} C_\xi$.

Let us verify that C is as desired. It is clear that $C \subseteq A$. Let $\xi < \nu$. There is $a_\xi \in A_{\alpha_\xi} - B_\xi$ such that

$$(\forall \beta \in \alpha_\xi \cap s_\xi^{-1}(\{1\})) (\forall k \in {}^\beta 2) \quad c_k \subseteq a_\xi.$$

Put $k_\xi = g \upharpoonright \alpha_\xi$. Then $a_\xi \in Z_{k_\xi}$ since $s_\xi(\beta) = f(\beta) = 1$ for every $\beta \in T \cap \alpha_\xi$. It follows that $c_{k_\xi} \in Z_{k_\xi}$. It is immediate that $Z_{k_\xi} = C_\xi$. Thus we have shown that (a) $C_\xi - B_\xi \neq \phi$ for every $\xi < \nu$, and (b) $c_{g \upharpoonright \alpha_\xi} \in C_\xi$ for every $\xi < \nu$. It follows from (a) that $C \in H^+$, and from (b) that $[C]_\kappa^2 = [C]_\prec^2$ since given $\xi, \zeta < \nu$ with $\alpha_\xi < \alpha_\zeta$, we have $c_{g \upharpoonright \alpha_\xi} \subseteq b$ for every $b \in C_\zeta$. \square

QUESTION. Is the assumption that κ is inaccessible necessary in the statement of Theorem 9.4 ?

Remark. Suppose κ is inaccessible. Then by Theorems 9.1, 9.2, 9.4, 5.4 and 4.7, $I_{\kappa, \lambda}$ is a weak χ -point if and only if $\lambda^{< \kappa} < \text{cov}(\mathbf{M}_{\kappa, \kappa})$ if and only if $\{C : [C]_\kappa^2 = [C]_\prec^2\}$ is dense in $(I_{\kappa, \lambda}^+, \subseteq)$.

10. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$

Definition. Let H be an ideal on $P_\kappa(\lambda)$ and α an ordinal. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$ means that given $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and F is identically 0 on $[B]_\kappa^2$ or (B, \prec) has order type α and F is identically 1 on $[B]_\kappa^2$.

In this section we show that $H^+ \xrightarrow{\kappa} (H^+, \omega + 1)^2$ for every ideal H on $P_\kappa(\lambda)$ with $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$.

Definition. Suppose that H is an ideal on $P_\kappa(\lambda), A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$. Then (H, A, F) is 0-good if there is $D \in H^+ \cap P(A)$ such that $\{b \in D : F(\cup(a \cap \kappa), b) = 1\} \in H$ for any $a \in D$.

The following is straightforward.

LEMMA 10.1. Suppose that (H, A, F) is 0-good, where H is an ideal on $P_\kappa(\lambda)$ which is both a weak π -point and a weak χ -point, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$. Then F is identically 0 on $[C]_\kappa^2$ for some $C \in H^+ \cap P(A)$.

Definition. Given an ideal H on $P_\kappa(\lambda)$ and $B \in H^+$, let $M_{H, B}^d$ be the set of all $Q \subseteq H^+ \cap P(B)$ such that (i) any two distinct members of Q are disjoint, and (ii) for every $A \in H^+ \cap P(B)$, there is $C \in Q$ with $A \cap C \in H^+$.

LEMMA 10.2. Suppose that (H, A, F) is not 0-good, where H is an ideal on $P_\kappa(\lambda)$, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$, and let $B \in H^+ \cap P(A)$. Then there exist $Q_B \in M_{H, B}^d$ and $\varphi_B : Q_B \rightarrow B$ such that (i) $\varphi_B(D) \prec b$ and $F(\cup(\varphi_B(D) \cap \kappa), b) = 1$ whenever $b \in D \in Q_B$, and (ii) $\cup(\varphi_B(D) \cap \kappa) \neq \cup(\varphi_B(D') \cap \kappa)$ for any two distinct members D and D' of Q_B .

Proof. Set $T = \{\cup(a \cap \kappa) : a \in B\}$ and define $\psi : T \times (H^+ \cap P(B)) \rightarrow P(B)$ by $\psi(\alpha, C) = \{b \in C : F(\alpha, b) = 1\}$. Now using induction, define $\eta \leq \kappa$ and $\alpha_\delta \in T$ and $B_\delta \in H^+ \cap P(B)$ for $\delta < \eta$ so that :

$$(0) \quad \text{If } \delta < \eta, B - (\bigcup_{\xi < \delta} B_\xi) \in H^+,$$

$$\alpha_\delta = \text{least } \alpha \in T \text{ such that } \psi(\alpha, B - (\bigcup_{\xi < \delta} B_\xi)) \in H^+$$

$$\text{and } B_\delta = \psi(\alpha_\delta, B - (\bigcup_{\xi < \delta} B_\xi)).$$

$$(1) \quad \text{If } \eta < \kappa, B - (\bigcup_{\xi < \eta} B_\xi) \in H.$$

Notice that if $\gamma < \delta < \eta$, then

$$\psi(\alpha_\delta, B - (\bigcup_{\xi < \delta} B_\xi)) \subseteq \psi(\alpha_\delta, B - (\bigcup_{\zeta < \delta} B_\zeta))$$

and consequently $\alpha_\gamma \leq \alpha_\delta$. In fact $\alpha_\gamma < \alpha_\delta$ as $\psi(\alpha_\gamma, B - (\bigcup_{\xi < \delta} B_\xi)) = \phi$ (since $(B - (\bigcup_{\xi < \delta} B_\xi)) \cap B_\gamma = \phi$)

$$\text{and } B_\gamma = \{b \in B - (\bigcup_{\zeta < \gamma} B_\zeta) : F(\alpha_\gamma, b) = 1\}.$$

We claim that $\{B_\delta : \delta < \eta\} \in M_{H,B}^d$. Suppose otherwise. Then there exists $E \in H^+ \cap P(B)$ such that $E \cap B_\xi \in H$ for every $\xi < \eta$. Since

$$E - (\bigcup_{\xi < \delta} B_\xi) \in H^+ \cap P(B - (\bigcup_{\xi < \delta} B_\xi))$$

for every $\delta < \kappa$, we must have $\eta = \kappa$. Set

$$\beta = \text{least } \alpha \in T \text{ such that } \psi(\alpha, E) \in H^+.$$

Then for each $\delta < \kappa$,

$$\psi(\beta, E) - (\bigcup_{\xi < \delta} B_\xi) \in H^+ \cap P(\psi(\beta, E) - (\bigcup_{\xi < \delta} B_\xi))$$

and therefore $\beta \geq \alpha_\delta$, which is a contradiction.

For each $\delta < \eta$, pick $s_\delta \in B$ so that $\cup(s_\delta \cap \kappa) = \alpha_\delta$, and put

$$S_\delta = \{b \in B_\delta : s_\delta \cup (\alpha_\delta + 2) \subseteq b\}.$$

Finally, set $Q_B = \{S_\delta : \delta < \eta\}$ and define $\varphi_B : Q_B \rightarrow B$ by $\varphi_B(S_\delta) = s_\delta$. \square

LEMMA 10.3. Suppose that H is an ideal on $P_\kappa(\lambda)$ and $A \in H^+$. Suppose further that $C \in H^+ \cap P(A)$ and $Q_\alpha \in M_{H,A}^d$ for $\alpha < \beta$, where β is a limit ordinal with $0 < \beta < \kappa$. Then

$$\{a \in C : (\forall h \in \prod_{\alpha < \beta} Q_\alpha) \quad a \notin \bigcap_{\alpha < \beta} h(\alpha)\} \in H.$$

Proof. It suffices to observe that for each $a \in \bigcap_{\alpha < \beta} (C \cap (\cup Q_\alpha))$, there is $h \in \prod_{\alpha < \beta} Q_\alpha$ such that $a \in \bigcap_{\alpha < \beta} h(\alpha)$. \square

LEMMA 10.4. Suppose that (H, A, F) is not 0-good, where H is an ideal on $P_\kappa(\lambda)$, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$. Then :

- (i) There is $C \subseteq A$ such that (C, \prec) has order type $\omega + 1$ and F is identically 1 on $[C]_\kappa^2$.
- (ii) Suppose that $\mathfrak{a}_H > \kappa$ and θ is uncountable cardinal $< \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$. Then there is $C \subseteq A$ such that (C, \prec) has order type $\theta + 1$ and F is identically 1 on $[C]_\kappa^2$.

Proof. We prove (ii) and leave the proof of (i) to the reader. By Corollary 19.7 in [EHMÁR], we have that $\mu^\tau < \kappa$ whenever μ and τ are cardinals such that $\theta \leq \mu < \kappa$ and $0 < \tau < \theta$. Using this and Lemmas 10.2 and 10.3, define $R_\beta, Q_\beta \in \{W \in M_{H,A}^d : |W| < \kappa\}$ and $\varphi_\beta : Q_\beta \rightarrow A$ for $\beta < \theta$ by :

$$(0) \quad R_0 = \{A\} ;$$

$$(1) \quad Q_\beta = \bigcup_{B \in R_\beta} Q_B ;$$

$$(2) \quad R_{\beta+1} = Q_\beta ;$$

$$(3) \quad R_\beta = H^+ \cap \{\bigcap_{\alpha < \beta} h(\alpha) : h \in \prod_{\alpha < \beta} Q_\alpha\} \text{ if } \beta \text{ is a limit ordinal} > 0 ;$$

$$(4) \quad \varphi_\beta = \bigcup_{B \in R_\beta} \varphi_B.$$

Select $b \in \bigcap_{\beta < \theta} (\cup Q_\beta)$. There must be $k \in \prod_{\beta < \theta} Q_\beta$ such that $b \in \bigcap_{\beta < \theta} k(\beta)$. Then

$$C = \{\varphi_\beta(k(\beta)) : \beta < \theta\} \cup \{b\}$$

is as desired. \square

THEOREM 10.5. Suppose θ is an infinite cardinal $< \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$. Then $H^+ \xrightarrow{\kappa} (H^+, \theta + 1)^2$ for every ideal H on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$.

Proof. Let H be an ideal on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$. Then H is a weak χ -point by Theorem 9.1. Moreover, H is a weak π -point and $\text{a}_H > \kappa$ by Theorem 8.4 since $\text{cov}(\mathbf{M}_{\kappa, \kappa}) \leq \mathfrak{d}_{\kappa, \lambda}^\kappa$ by Proposition 4.1. Hence, $H^+ \xrightarrow{\kappa} (H^+, \theta + 1)^2$ by Lemmas 10.1 and 10.4. \square

11. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$

Definition. For $A \subseteq P_\kappa(\lambda)$, let

$$[A]_{\kappa, \kappa}^2 = \{(\cup(a \cap \kappa), \cup(b \cap \kappa)) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}$$

Remark. $[P_\kappa(\lambda)]_{\kappa, \kappa}^2 = [\kappa]^2$.

Definition. Let H be an ideal on $P_\kappa(\lambda)$ and α an ordinal. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$ means that given $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and F is identically 0 on $[B]_{\kappa, \kappa}^2$, or (B, \prec) has order type α and F is identically 1 on $[B]_{\kappa, \kappa}^2$.

We will show that $H^+ \xrightarrow{\kappa} (H^+, \omega + 1)^2$ for every ideal H on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{non}_\kappa$ (weakly selective).

Definition. For an ideal H on $P_\kappa(\lambda)$, $J_H = \{B \subseteq \kappa : U_B \in H\}$, where

$$U_B = \{a \in P_\kappa(\lambda) : \cup(a \cap \kappa) \in B\}.$$

LEMMA 11.1. Let H be an ideal on $P_\kappa(\lambda)$. Then J_H is an ideal on κ . Moreover, $\text{cof}(J_H) \leq \text{cof}(H)$.

Proof. It is simple to see that (a) $U_\kappa = P_\kappa(\lambda)$, (b) $U_{\cup \mathfrak{B}} \subseteq \bigcup_{B \in \mathfrak{B}} U_B$ for $\mathfrak{B} \subseteq P(\kappa)$, (c) $U_C \subseteq U_B$ if $C \subseteq B \subseteq \kappa$, and (d) $U_B \in I_{\kappa, \lambda}$ for every $B \subseteq \kappa$ with $|B| = 1$. The first assertion immediately follows.

For $C \subseteq P_\kappa(\lambda)$, let Y_C be the set of all $\delta \in \kappa$ such that

$$\{a \in P_\kappa(\lambda) : \cup(a \cap \kappa) = \delta\} \subseteq C.$$

If $C \in H$, then $Y_C \in J_H$ since $U_{Y_C} \subseteq C$. Moreover if $B \subseteq \kappa$ and $U_B \subseteq C \subseteq P_\kappa(\lambda)$, then $B \subseteq Y_C$. Hence $\text{cof}(J_H) \leq \text{cof}(H)$. \square

Remark. Let H be an ideal on $P_\kappa(\lambda)$. Then

$$\{\cup(a \cap \kappa) : a \in A\} \in (J_{H|A})^+$$

for every $A \in H^+$.

The following is readily checked.

LEMMA 11.2. Given an ideal H on $P_\kappa(\lambda)$, the following are equivalent :

- (i) J_H is a local Q -point.
- (ii) For every $g \in {}^\kappa\kappa$, there is $B \in H^+$ such that $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

Suppose κ is a limit cardinal. If $\kappa^+ < \text{non}_\kappa(\text{weak } Q\text{-point})$, then by Lemma 11.1 $J_{I_{\kappa,\kappa^+}|A}$ is a local Q -point for every $A \in I_{\kappa,\kappa^+}^+$. The following shows that this implication can be reversed.

PROPOSITION 11.3. Suppose that κ is a limit cardinal and $J_{I_{\kappa,\lambda}|A}$ is a local Q -point for every $A \in I_{\kappa,\lambda}^+$. Then $\sigma < \overline{\text{non}}_\kappa(\text{weak } Q\text{-point})$ for every $\sigma \in \mathcal{K}(\kappa, \lambda)$.

Proof. Suppose that J is an ideal on κ and $T \subseteq P_\kappa(\lambda - \kappa)$ is such that $\overline{\text{cof}}(J) \leq |T|$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. Select $B_d \in J$ for $d \in T$ so that for every $D \in J$, there is $u \in P_\kappa(T) - \{\phi\}$ with $D \subseteq \bigcup_{d \in u} B_d$. Let A be the set of all $a \in P_\kappa(\lambda)$ such that $\cup(a \cap \kappa) \notin B_d$ for every $d \in T \cap P(a - \kappa)$.

It is simple to see that $A \in I_{\kappa,\lambda}^+$. Now fix $g \in {}^\kappa\kappa$. By Lemma 11.2, there is $C \in (I_{\kappa,\lambda} | A)^+$ such that $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$ for all $a, b \in C$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Set

$$D = \{\cup(a \cap \kappa) : a \in C \cap A\}.$$

Then $D \in J^+$. Moreover $g(\alpha) < \beta$ for all $\alpha, \beta \in D$ with $\alpha < \beta$. Hence J is a local Q -point. \square

THEOREM 11.4. Suppose that θ is an infinite cardinal $< \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$, and H is an ideal on $P_\kappa(\lambda)$ with $\text{cof}(H) < \text{non}_\kappa(\text{weakly selective})$. Then $H^+ \xrightarrow{\kappa} (H^+, \theta + 1)^2$.

Proof. Fix $G : \kappa \times \kappa \rightarrow 2$ and $A \in H^+$. Define $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ by $F(\alpha, b) = G(\alpha, \cup(b \cap \kappa))$.

First suppose (H, A, F) is 0-good. Pick $D \in H^+ \cap P(A)$ so that

$$\{b \in D : F(\cup(a \cap \kappa), b) = 1\} \in H$$

for any $a \in D$. Set $B_\alpha = \{\delta < \kappa : G(\alpha, \delta) = 1\}$ for $\alpha < \kappa$. Then $B_{\cup(a \cap \kappa)} \in J_{H|D}$ for every $a \in D$ since

$$D \cap U_{B_{\cup(a \cap \kappa)}} = \{b \in D : G(\cup(a \cap \kappa), \cup(b \cap \kappa)) = 1\} = \{b \in D : F(\cup(a \cap \kappa), b) = 1\}.$$

By Lemma 11.1 $\text{cof}(J_{H|D}) < \text{non}_\kappa(\text{weak } P\text{-point})$ so there is $G \in (J_{H|D})^+$ such that $|G \cap B_{\cup(a \cap \kappa)}| < \kappa$ for every $a \in D$. Notice that $D \cap U_G \in H^+$. Select $g \in {}^\kappa\kappa$ so that $\cup(b \cap \kappa) \notin B_{\cup(a \cap \kappa)}$ for all $a, b \in D \cap U_G$ such that $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$. By Lemma 11.1

$$\text{cof}(J_{H|(D \cap U_G)}) < \text{non}_\kappa(\text{weak } Q\text{-point})$$

and hence by Lemma 11.2 there is $R \in (H | (D \cap U_G))^+$ such that $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$ for all $a, b \in R$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Then $R \cap D \cap U_G \in H^+ \cap P(A)$ and moreover F is identically 0 on $[R \cap D \cap U_G]_{\kappa, \kappa}^2$.

Finally, suppose (H, A, F) is not 0-good. Since $\alpha_H > \kappa$ by Theorems 2.2 and 8.4, there is by Lemma 10.4 $C \subseteq A$ such that (C, \prec) has order type $\theta + 1$ and F is identically 1 on $[C]_\kappa^2$. It is immediate that G is constantly 1 on $[C]_{\kappa, \kappa}^2$. \square

Remark. Suppose κ is a successor cardinal. Then by Theorem 11.4 $\kappa^+ < \aleph_\kappa$ implies that $I_{\kappa, \kappa^+}^+ \xrightarrow{\kappa} (I_{\kappa, \kappa^+}^+, \theta + 1)^2$ for every cardinal $\theta \geq 2$ such that $\kappa \rightarrow (\kappa, \theta)^2$. Conversely, it will be shown in the next section that $I_{\kappa, \kappa^+}^+ \xrightarrow{\kappa} (I_{\kappa, \kappa^+}^+, 3)^2$ implies that $\kappa^+ < \aleph_\kappa$.

12. $H^+ \xrightarrow{\kappa} (H^+; \alpha)^2$

Definition. Given an ideal H on $P_\kappa(\lambda)$ and an ordinal α , $H^+ \xrightarrow{\kappa} (H^+; \alpha)^2$ means that for all $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and F is identically 0 on $[B]_{\kappa, \kappa}^2$, or $\{\cup(a \cap \kappa) : a \in B\}$ has order type α and F is identically 1 on $[B]_{\kappa, \kappa}^2$.

Remark. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow{\kappa} (H^+; \alpha)^2 \Rightarrow \kappa \rightarrow (\kappa, \alpha)^2$.

We will prove that $I_{\kappa, \kappa}^+ \xrightarrow{\kappa} (I_{\kappa, \kappa^+}^+; \alpha)^2$ if and only if $\kappa^+ < \text{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$.

THEOREM 12.1. Suppose that $3 \leq \alpha \leq \kappa$ and H is an ideal on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$. Then $H^+ \xrightarrow{\kappa} (H^+; \alpha)^2$.

Proof. By Lemma 11.1, $(J_{H|A})^+ \rightarrow ((J_{H|A})^+, \alpha)^2$ for every $A \in H^+$. The desired conclusion easily follows. \square

THEOREM 12.2. Suppose that $3 \leq \alpha \leq \kappa$ and $I_{\kappa, \lambda}^+ \xrightarrow{\kappa} (I_{\kappa, \lambda}^+; \alpha)^2$. Then $\sigma < \overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ for every $\sigma \in \mathcal{K}(\kappa, \lambda)$.

Proof. The proof is an easy modification of that of Proposition 11.3. \square

Remark. Suppose that κ is inaccessible and $3 \leq \alpha \leq \kappa$. Then by Theorems 12.1 and 12.2, $I_{\kappa, \lambda}^+ \xrightarrow{\kappa} (I_{\kappa, \lambda}^+; \alpha)^2$ if and only if $\lambda^{<\kappa} < \text{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$.

Let us finally observe that for $3 \leq \alpha \leq \kappa$, there always exists an ideal H on $P_\kappa(\lambda)$ of the least possible cofinality such that $H^+ \xrightarrow{\kappa} (H^+; \alpha)^2$:

PROPOSITION 12.3. Given $3 \leq \alpha \leq \kappa$, there is an ideal H on $P_\kappa(\lambda)$ such that (a) $H^+ \xrightarrow{\kappa} (H^+; \alpha)^2$, (b) $\text{cof}(H) = u(\kappa, \lambda) \cdot \text{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$, and (c) $\overline{\text{cof}}(H) \leq \lambda \cdot \overline{\text{non}}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$.

Proof. Argue as for Lemma 5.1 of [M2]. \square

13. $H^+ \xrightarrow{\kappa} (H^+)^2$

Definition. Given an ideal H on $P_\kappa(\lambda)$, $H^+ \xrightarrow{\kappa} (H^+)^2$ (respectively, $H^+ \xrightarrow{\kappa} (H^+)^2$) means that for all $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow 2$ (respectively, $F : [P_\kappa(\lambda)]_{\kappa,\kappa}^2 \rightarrow 2$) and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that F is constant on $[B]_\kappa^2$ (respectively, $[B]_{\kappa,\kappa}^2$).

THEOREM 13.1. Suppose κ is weakly compact. Then $H^+ \xrightarrow{\kappa} (H^+)^2$ for every ideal H on $P_\kappa(\lambda)$ such that $cof(H) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$.

Proof. Suppose that H is an ideal on $P_\kappa(\lambda)$ with $cof(H) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$, $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ and $A \in H^+$. Then $cof(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$ by Proposition 4.1 and therefore by a result of [M5] there are $B \in H^+ \cap P(A)$ and $i < 2$ such that

$$\{b \in B : F(\cup(a \cap \kappa), b) \neq i\} \in I_{\kappa,\lambda}$$

for every $a \in B$. Since H is a weak χ -point by Theorem 9.1, there is $C \in H^+ \cap P(B)$ such that F takes the constant value i on $[C]_\kappa^2$. \square

Remark. It follows from Theorem 6.5 (ii) and Theorem 15.1 (below) that if κ is weakly compact, then $H^+ \xrightarrow{\kappa} (H^+)^2$ for every ideal H on $P_\kappa(\lambda)$ such that $cof(H) < \text{non}_\kappa(\text{weakly selective})$.

COROLLARY 13.2. The following are equivalent :

- (i) κ is weakly compact and $\lambda^{<\kappa} < \text{cov}(\mathbf{M}_{\kappa,\kappa})$.
- (ii) $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$.
- (iii) $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+; \kappa)^2$.

Proof. (i) \rightarrow (ii) : By Theorem 13.1.

(ii) \rightarrow (iii) : Trivial.

(iii) \rightarrow (i) : By Theorems 12.2, 6.5 (i), 6.1 (iii), 5.4 and 4.7. \square

14. $H^+ \xrightarrow{\kappa} [H^+]_\rho^2$

Definition. Given a cardinal ρ with $2 \leq \rho \leq \lambda^{<\kappa}$ and an ideal H on $P_\kappa(\lambda)$, $H^+ \xrightarrow{\kappa} [H^+]_\rho^2$ means that for all $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow \rho$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $F''[B]_\kappa^2 \neq \rho$.

THEOREM 14.1. Suppose that κ is a limit cardinal and H is an ideal on $P_\kappa(\lambda)$ such that $cof(H) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$. Then $H^+ \xrightarrow{\kappa} [H^+]_{\kappa^+}^2$.

Proof. Fix $F : \kappa \times P_\kappa(\lambda) \rightarrow \kappa^+$ and $A \in H^+$. Since $cof(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$ by Proposition 4.1, there are $B \in H^+ \cap P(A)$ and $\xi \in \kappa^+$ such that $\{b \in B : F(\cup(a \cap \kappa), b) = \xi\} \in I_{\kappa,\lambda}$ for every $a \in B$ ([M5]). Now H is a weak χ -point by Theorem 9.1 and so $\xi \notin F''[C]_\kappa^2$ for some $C \in H^+ \cap P(B)$. \square

Let us now show that $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$ if $\lambda \geq \bar{\mathfrak{d}}_\kappa$. We will need some definitions.

Definition. Given $f \in \prod_{\alpha \in \kappa} (\kappa - \alpha)$, we define $\tilde{f} \in {}^\kappa \kappa$ by stipulating that

- (i) $\tilde{f}(0) = 0$;
- (ii) $\tilde{f}(\xi + 1) = f(\tilde{f}(\xi)) + 1$;
- (iii) $\tilde{f}(\xi) = \bigcup_{\zeta < \xi} \tilde{f}(\zeta)$ if ξ is a limit ordinal > 0 .

Remark. \tilde{f} is a strictly increasing function.

Remark. If $g \in {}^\kappa\kappa$ is a strictly increasing function such that $g(\alpha) \leq f(\alpha)$ for all $\alpha < \kappa$, then $g(\tilde{f}(\xi)) \in [\tilde{f}(\xi), \tilde{f}(\xi + 1))$ for every $\xi < \kappa$.

Definition. Given $f \in \prod_{\alpha \in \kappa} (\kappa - \alpha)$ and a cardinal $\tau \in (0, \kappa)$, we define $c_{f,\tau} : \tilde{f}(\tau) \rightarrow \tau$ by stipulating that $c_{f,\tau}$ takes the constant value ξ on $[\tilde{f}(\xi), \tilde{f}(\xi + 1))$.

Definition. Suppose that $T \subseteq P_\kappa(\lambda - \kappa)$ is such that (a) $|T| \geq \bar{\delta}_\kappa$, and (b) $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

Let $\psi_T : T \rightarrow {}^\kappa\kappa$ be such that given $g \in {}^\kappa\kappa$, there is $u \in P_\kappa(T) - \{\phi\}$ such that

$$g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$$

for all $\alpha < \kappa$.

For $e \in P_\kappa(\lambda - \kappa)$, let $\tau_{T,e} = |T \cap P(e)|$ and select a bijection $k_{T,e} : \tau_{T,e} \rightarrow T \cap P(e)$.

Also, define $f_{T,e} \in {}^\kappa\kappa$ by

$$f_{T,e}(\alpha) = \max(\alpha, \bigcup_{d \in T \cap P(e)} (\psi_T(d))(\alpha)).$$

Finally, let A_T be the set of all $a \in P_\kappa(\lambda)$ such that (i) $T \cap P(a - \kappa) \neq \phi$, and (ii) $\cup(a \cap \kappa) \geq \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$.

Remark. $A_T \in I_{\kappa,\lambda}^+$.

THEOREM 14.2. Suppose that $\rho \in \mathcal{K}(\kappa, \lambda)$ and $\rho \geq \bar{\delta}_\kappa$. Then $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\rho^2$.

Proof. Select $T \subseteq P_\kappa(\lambda - \kappa)$ so that $|T| = \rho$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. We define a partial function F from $\kappa \times A_T$ to T by stipulating that

$$F(\beta, a) = k_{T,a-\kappa}(c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}(\beta))$$

if $a \in A_T$ and $\beta < \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$.

Now fix $B \in I_{\kappa,\lambda}^+ \cap P(A_T)$ and $x \in T$. Let $g \in {}^\kappa\kappa$ be the increasing enumeration of the elements of the set $\{\cup(b \cap \kappa) : b \in B\}$. Select $u \in P_\kappa(T) - \{\phi\}$ so that $g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$ for all $\alpha < \kappa$. Now pick $a \in B$ so that $x \cup (\cup u) \subseteq a$. Notice that $g(\alpha) \leq f_{T,a-\kappa}(\alpha)$ for every $\alpha < \kappa$. Let $\xi \in \tau_{T,a-\kappa}$ be such that $k_{T,a-\kappa}(\xi) = x$. Then

$$\tilde{f}_{T,a-\kappa}(\xi) \leq g(\tilde{f}_{T,a-\kappa}(\xi)) < \tilde{f}_{T,a-\kappa}(\xi + 1) \leq \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa}) \leq \cup(a \cap \kappa).$$

Moreover,

$$F(g(\tilde{f}_{T,a-\kappa}(\xi)), a) = k_{T,a-\kappa}(\xi) = x.$$

since

$$c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}(g(\tilde{f}_{T,a-\kappa}(\xi))) = \xi$$

□

15. $H^+ \xrightarrow[\kappa]{} [H^+]_\rho^2$

Definition. Given a cardinal $\rho \in [2, \kappa]$ and an ideal H on $P_\kappa(\lambda)$, $H^+ \xrightarrow[\kappa]{} [H^+]_\rho^2$ means that for all $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow \rho$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $F''[B]_{\kappa, \kappa}^2 \neq \rho$.

Remark. $\kappa \not\rightarrow [\kappa]_\rho^2 \Rightarrow H^+ \xrightarrow[\kappa]{} [H^+]_\rho^2 \Rightarrow H^+ \not\xrightarrow[\kappa]{} [H^+]_\rho^2$.

The following result shows that $I_{\kappa, \kappa^+}^+ \xrightarrow[\kappa]{} [I_{\kappa, \kappa^+}]_\rho^2$ if and only if $\kappa^+ < \text{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$.

THEOREM 15.1. Let ρ be a cardinal with $2 \leq \rho \leq \kappa$. Then :

- (i) $H^+ \xrightarrow[\kappa]{} [H^+]_\rho^2$ for every ideal H on $P_\kappa(\lambda)$ such that $\text{cof}(H) < \text{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$.
- (ii) If $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{} [I_{\kappa, \lambda}]_\rho^2$, then $\sigma < \overline{\text{non}}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$ for every $\sigma \in \mathcal{K}(\kappa, \lambda)$.

Proof. (i) : Use Lemma 11.1.

(ii) : Argue as for Proposition 11.3. □

Remark. Thus assuming κ is inaccessible, $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{} [I_{\kappa, \lambda}]_\rho^2$ if and only if $\lambda^{<\kappa} < \text{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$.

Finally, we show that if $\lambda \geq \overline{\mathfrak{d}}_\kappa$ and κ is a limit cardinal such that $2^{<\kappa} = \kappa$, then $I_{\kappa, \lambda}^+ \not\xrightarrow[\kappa]{} [I_{\kappa, \lambda}]_\kappa^2$.

THEOREM 15.2. Suppose that (a) κ is a limit cardinal such that $2^{<\kappa} = \kappa$, and (b) either $\lambda > \overline{\mathfrak{d}}_\kappa$, or $\overline{\mathfrak{d}}_\kappa \in \mathcal{K}(\kappa, \lambda)$. Then $I_{\kappa, \lambda}^+ \not\xrightarrow[\kappa]{} [I_{\kappa, \lambda}]_\kappa^2$.

Proof. Select $T \subseteq P_\kappa(\lambda - \kappa)$ so that $|T| = \lambda \cdot \overline{\mathfrak{d}}_\kappa$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. Also, select $\chi : \kappa \rightarrow \bigcup_{\gamma < \kappa} \gamma_\kappa$ so that $|\chi^{-1}(\{z\})| = \kappa$ for every $z \in \bigcup_{\gamma < \kappa} \gamma_\kappa$. Now let A be the set of all $a \in A_T$ such that

$$\chi(\cup(a \cap \kappa)) = c_{f_{T, a-\kappa}, \tau_{T, a-\kappa}}.$$

Notice that $A \in I_{\kappa, \lambda}^+$. We define a partial function F from $\kappa \times \kappa$ to κ by stipulating that $F(\delta, \eta) = (\chi(\eta))(\delta)$ if $\eta \in \kappa$ and $\delta \in \text{dom}(\chi(\eta))$.

Now fix $B \in I_{\kappa, \lambda}^+ \cap P(A)$ and $\xi \in \kappa$. Let $g \in {}^\kappa \kappa$ be the increasing enumeration of the elements of the set $\{\cup(b \cap \kappa) : b \in B\}$. Select $u \in P_\kappa(T) - \{\phi\}$ so that $g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$ for all $\alpha < \kappa$. Pick $a \in B$ so that $\cup u \subseteq a$ and $|T \cap P(a)| > \xi$. Then

$$g(\tilde{f}_{T, a-\kappa}(\xi)) < \cup(a \cap \kappa)$$

and

$$\xi = c_{f_{T, a-\kappa}, \tau_{T, a-\kappa}}(g(\tilde{f}_{T, a-\kappa}(\xi))) = (\chi(\cup(a \cap \kappa)))(g(\tilde{f}_{T, a-\kappa}(\xi))) = F(g(\tilde{f}_{T, a-\kappa}(\xi)), \cup(a \cap \kappa)).$$

□

Remark. Theorems 14.2, 15.1 and 15.2 (as well as e.g. Theorems 9.2, 9.4, 12.1 and 12.2, Propositions 9.3 and 11.3 and Corollary 13.2) are also true for $\kappa = \omega$. This gives (a) $\mathfrak{d} \geq \text{non}_\omega(J^+ \rightarrow [J^+]_\omega^2)$, and (b) if $\lambda \geq \mathfrak{d}$, then $I_{\omega, \lambda}^+ \not\xrightarrow[\omega]{} [I_{\omega, \lambda}]_\lambda^2$ and $I_{\omega, \lambda}^+ \not\xrightarrow[\omega]{} [I_{\omega, \lambda}]_\omega^2$.

REFERENCES

- [A] **Y. ABE** - *Seminormal fine measures on $P_\kappa(\lambda)$* ; in : Proceedings of the Sixth Asian Logic Conference, (C.T. Chong et al., eds.), World Scientific, Singapore, 1998, pp. 1-12.
- [B] **T. BARTOSZYŃSKI** - *Combinatorial aspects of measure and category* ; Fundamenta Mathematicae 127 (1987), 225-239.
- [Bau1] **J.E. BAUMGARTNER** - *Ineffability properties of cardinals I* ; in : Infinite and Finite Sets (A. Hajnal, R. Rado and V.T. Sós, eds.), Colloquia Mathematica Societatis János Bolyai vol. 10, North-Holland, Amsterdam, 1975, pp. 109-130.
- [Bau2] **J.E. BAUMGARTNER** - *Ineffability properties of cardinals II* ; in : Logic, Foundations of Mathematics and Computability Theory (R.E. Butts and J. Hintikka, eds.), The University of Western Ontario Series in Philosophy and Science vol. 9, Reidel, Dordrecht (Holland), 1977, pp. 87-106.
- [BauT] **J.E. BAUMGARTNER and A.D. TAYLOR** - *Partition theorems and ultrafilters* ; Transactions of the American Mathematical Society 241 (1978), 283-309.
- [BauTW] **J.E. BAUMGARTNER, A.D. TAYLOR and S. WAGON** - *Structural properties of ideals* ; Dissertationes Mathematicae (Rozprawy Matematyczne) 197 (1982), 1-95.
- [Bl] **A. BLASS** - *Ultrafilter mappings and their Dedekind cuts* ; Transactions of the American Mathematical Society 188 (1974), 327-340.
- [CFMag] **J. CUMMINGS, M. FOREMAN and M. MAGIDOR** - *Squares, scales and stationary reflection* ; Journal of Mathematical Logic 1 (2001), 35-98.
- [CS] **J. CUMMINGS and S. SHELAH** - *Cardinal invariants above the continuum* ; Annals of Pure and Applied Logic 75 (1995), 251-268.
- [DMi] **B. DUSHNIK and E.W. MILLER** - *Partially ordered sets* ; American Journal of Mathematics 63 (1941), 600-610.
- [EHMáR] **P. ERDÖS, A. HAJNAL, A. MÁTÉ and R. RADO** - *Combinatorial Set Theory : Partition Relations for Cardinals* ; Studies in Logic and the Foundations of Mathematics vol. 106, North-Holland, Amsterdam, 1984.
- [ER] **P. ERDÖS and R. RADO** - *A partition calculus in set theory* ; Bulletin of the American Mathematical Society 62 (1956), 427-489.
- [Ka] **A. KANAMORI** - *The Higher Infinite* ; Perspectives in Mathematical Logic, Springer, Berlin, 1994.
- [K] **K. KUNEN** - *Set Theory* ; North-Holland, Amsterdam, 1980.
- [Laf] **C. LAFLAMME** - *Strong meager properties for filters* ; Fundamenta Mathematicae 146 (1995), 283-293.
- [L1] **A. LANDVER** - *Singular Baire numbers and related topics* ; Ph. D. Thesis, University of Wisconsin, Madison, Wisconsin, 1990.
- [L2] **A. LANDVER** - *Baire numbers, uncountable Cohen sets and perfect-set forcing* ; Journal of Symbolic Logic 57 (1992), 1086-1107.
- [M1] **P. MATET** - *Combinatorics and forcing with distributive ideals* ; Annals of Pure and Applied Logic 86 (1997), 137-201.

- [M2] **P. MATET** - *The covering number for category and partition relations on $P_\omega(\lambda)$* ; Fundamenta Mathematicae 171 (2002), 235-247.
- [M3] **P. MATET** - *Partition relations for κ -normal ideals on $P_\kappa(\lambda)$* ; Annals of Pure and Applied Logic 121 (2003), 89-111.
- [M4] **P. MATET** - *A partition property of a mixed type for $P_\kappa(\lambda)$* ; Mathematical Logic Quarterly 49 (2003), 615-628.
- [M5] **P. MATET** - *Weak square bracket relations for $P_\kappa(\lambda)$* ; in preparation.
- [M6] **P. MATET** - *Covering for category and combinatorics on $P_\kappa(\lambda)$* ; preprint.
- [MP1] **P. MATET and J. PAWLIKOWSKI** - *Ideals over ω and cardinal invariants of the continuum* ; Journal of Symbolic Logic 63 (1998), 1040-1054.
- [MP2] **P. MATET and J. PAWLIKOWSKI** - *Q -pointness, P -pointness and feebleness of ideals* ; Journal of Symbolic Logic 68 (2003), 235-261.
- [MPé] **P. MATET and C. PÉAN** - *Distributivity properties on $P_\omega(\lambda)$* ; Discrete Mathematics, to appear.
- [MPéS1] **P. MATET, C. PÉAN and S. SHELAH** - *Cofinality of normal ideals on $P_\kappa(\lambda)$ I* ; preprint.
- [MPéS2] **P. MATET, C. PÉAN and S. SHELAH** - *Cofinality of normal ideals on $P_\kappa(\lambda)$ II* ; preprint.
- [MROs] **P. MATET, A. ROSŁANOWSKI and S. SHELAH** - *Cofinality of the nonstationary ideal* ; preprint.
- [Mil1] **A.W. MILLER** - *A characterization of the least cardinal for which the Baire category theorem fails* ; Proceedings of the American Mathematical Society 86 (1982), 498-502.
- [Mil2] **A.W. MILLER** - *The Baire category theorem and cardinals of countable cofinality* ; Journal of Symbolic Logic 47 (1982), 275-288.
- [S] **S. SHELAH** - *On successors of singular cardinals* ; in : Logic Colloquium'78 (M. Boffa et al., eds.), North-Holland, Amsterdam, 1979, pp. 357-380.
- [To1] **S. TODORCEVIC** - *Partitioning pairs of countable ordinals* ; Acta Mathematica 159 (1987), 261-294.
- [To2] **S. TODORCEVIC** - *Coherent sequences* ; in : Handbook of Set Theory (M. Foreman, A. Kanamori and M. Magidor, eds.), Kluwer, Dordrecht (Holland), to appear.

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